Journal of Statistical Physics, Vol. 23, No. 1, 1980

Microscopic Modes in a Fermi Superfluid. I. Linearized Kinetic Equations

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Received March 12, 1979

This is the first of two papers in which microscopic expressions for the amplitudes and dispersion relations for hydrodynamic modes in an isotropic Fermi superfluid are derived. In this first paper we derive closed, decoupled, linearized kinetic equations for the bogolon spin density and total density in a Fermi superfluid with fluctuating superfluid velocity, and we discuss the form of the hydrodynamic equations that result from these equations.

KEY WORDS: Fermi superfluids; transport theory; hydrodynamic modes; microscopic mode theory; broken symmetry; Wigner functions.

1. INTRODUCTION

Hydrodynamic modes characterize the low-frequency, long-wavelength behavior of macroscopic systems. They may originate either from collisional invariants or broken symmetries in a system. In the first case, hydrodynamic behavior results because collisions cannot smooth out inhomogeneities in the densities of collisional invariants. Such quantities must be transported across the fluid to achieve equilibrium. Hydrodynamic behavior also arises when symmetries are broken at a phase transition. This can happen when fluctuations in some quantity (such as spin orientation in a magnetic crystal or the phase of the macroscopic wave function in a superconductor) become correlated over infinitely long distances. The equilibrium state is then characterized by an additional thermodynamic variable, the order parameter. It may happen that inhomogeneities in a given order parameter decay very slowly because collisions cannot destroy them. Then, information about the inhomogeneity must be transmitted from one part of the fluid to another, and inhomogeneities in the order parameter behave very much like inhomogeneities in the densities of conserved quantities.

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0022-4715/80/0700-0083$03.00/0

One example of a system which contains both types of hydrodynamic modes is a Fermi superfluid. In this system, fermions form bound pairs which condense into a single quantum state. This condensed phase may be characterized by a macroscopic wave function. If the phase of this wave function varies in space and time, then a new hydrodynamic equation results for the velocity of the condensed phase (superfluid velocity). The condensed phase will accelerate if there is a gradient in the chemical potential. However, gradients in the chemical potential are related to gradients in the pressure and temperature via the Gibbs–Duheim equation. Since pressure and temperature involve collisional invariants, spatial variations in the superfluid velocity decay via transport processes.

If one wishes to obtain the dispersion relations for the hydrodynamic modes in a system, the traditional method is to find the normal mode frequencies of the linear hydrodynamic equations. The normal mode frequencies will depend on various equilibrium response functions and the transport coefficients. One must then compute the response functions using a microscopic theory and compute the transport coefficients using Chapman–Enskog theory, in order to obtain microscopic expressions for the dispersion relations. However, it has been shown by Résibois[1] (for the case of classical systems) and by the author[2] (for the case of normal Fermi liquids) that microscopic expressions for the dispersion relations and amplitudes of hydrodynamic modes can be obtained directly from the linearized kinetic equation by using perturbation theory. Furthermore, by equating the microscopic dispersion relations to the macroscopic dispersion relations obtained from the hydrodynamic equations, one can also obtain microscopic expressions for the transport coefficients. This method focuses on the kinetic equation itself and not on the hydrodynamic equations, as does the Chapman–Enskog approach, and is very sensitive to the detailed properties of the kinetic equation.

The microscopic mode theory has not yet been applied to a system in which part of the hydrodynamic behavior comes from a broken symmetry. But as we shall show, it can be used for such systems and gives rise to the dispersion relations for the modes due to broken symmetry in a very interesting way. In order to keep our calculations as simple as possible, we will discuss the hydrodynamic behavior of a Fermi superfluid whose dynamics is given by the Gor'kov Hamiltonian. The kinetic equation we shall use can be derived by a very elegant method due to Peletminskii and Yatsenko.[3] This method was first applied to Fermi superfluids by Galaiko,[4] who derived the kinetic equations for a homogeneous Fermi superfluid, using the Gor'kov model. Thus, Galaiko's equations describe a system with spherical, spinless pairs. The work of Galaiko was later generalized to the case of an inhomogeneous system by Shumeiko,[5] who then used a Chapman–Enskog type of approach to derive expressions for transport coefficients. The analysis of Galaiko and Shumeiko is not useful when one wishes to derive the microscopic modes directly from the kinetic equation. The kinetic equation must contain information about the mode due to broken gauge symmetry; and it must be closed. An analysis which does lead to closure of the kinetic equation has been given by Betbeder-Matibet and Nozières[6] for a collisionless Fermi superfluid in the presence of a spatially varying external field. We shall follow their approach here.

One interesting feature of a Fermi superfluid is that the hydrodynamics cannot be discussed in terms of particle parameters, but must be discussed in terms of the basic excitations, which we call bogolons. It is the properties of bogolons that are conserved during collisions, and not particles. In this first paper we will derive the kinetic equations for both the bogolon spin density and the total bogolon density, and we will derive and discuss the properties of the collision operators associated with each of these equations. The collision operators are the key to the hydrodynamic behavior in a system. Although we have tried to make these papers self-contained, we shall focus on those aspects of the derivation that are important for understanding the physics involved or the detailed form of the equations.

We shall begin in Section 2 with a discussion of the basic model, and we shall obtain an expression for the kinetic "matrix" which describes particle propagation in the lab frame in terms of particle Wigner functions and pair densities referred to the superfluid rest frame. In Section 3, we shall linearize the kinetic matrix and transform it to a form which describes bogolon propagation. Using the method of Betbeder-Matibet and Nozières, we can then write the bogolon kinetic equations in scalar form and to a large extent (but not completely) close them. In Section 4, we derive an expression for the linearized bogolon collision integral and show that the bogolon kinetic equations decouple into two kinetic equations, one for the bogolon spin density and another for the total bogolon density. The collision operator for the total bogolon density conserves bogolon momentum and energy, thus leading to four hydrodynamic equations (no continuity equation). The collision operator for the bogolon spin density only conserves a constant, thus leading to the bogolon spin diffusion equation. For a system described by these kinetic equations, bogolon number is not conserved.

In order to close the kinetic equations, in Section 5 the macroscopic phase is determined in such a way as to ensure particle conservation. This is now a standard procedure and completely closes the kinetic equation. Finally, in Section 6, for completeness and for later use, we shall derive the standard linearized two-fluid hydrodynamic equations from the kinetic equations, and discuss the form of the hydrodynamic modes obtained from the linearized hydrodynamic equations.

In a subsequent paper we shall derive microscopic expressions for the hydrodynamic modes directly from the kinetic equations, using perturbation theory.

2. KINETIC EQUATION

Let us consider a Fermi system whose dynamics is governed by a Hamiltonian of the form

$$\hat{H} = \sum_{\sigma=\uparrow,\downarrow} \int d\mathbf{r}\, \hat{\Psi}_\sigma^{+}(\mathbf{r})\left(\frac{-\hbar^2}{2m}\nabla_{\mathbf{r}}^2 - \mu^0\right)\hat{\Psi}_\sigma(\mathbf{r})$$
$$+ \iint d\mathbf{r}_1\, d\mathbf{r}_2\, V(|\mathbf{r}_1 - \mathbf{r}_2|)\hat{\Psi}_\uparrow^{+}(\mathbf{r}_1)\hat{\Psi}_\downarrow^{+}(\mathbf{r}_2)\hat{\Psi}_\downarrow(\mathbf{r}_2)\Psi_\uparrow(\mathbf{r}_1) \tag{2.1}$$

Only particles with antiparallel spin interact. We assume the interaction is spherically symmetric (we will specialize to the Gor'kov case later). The quantities $\hat{\Psi}_\sigma^{+}(\mathbf{r})$ and $\hat{\Psi}_\sigma(\mathbf{r})$ are the field operators for particles of spin σ, and μ^0 is the chemical potential of the equilibrium system. The field operators obey fermion anticommutation relations.

The density operator for the system is governed by the equation of motion

$$\partial\hat{\rho}/\partial t = -(i/\hbar)[\hat{H}, \hat{\rho}(t)] \tag{2.2}$$

We introduce a one-body reduced density matrix $\bar{N}(\mathbf{r}_1, \mathbf{r}_2)$, which obeys the equation of motion

$$\frac{\partial \bar{N}(\mathbf{r}_1, \mathbf{r}_2, t)}{\partial t} = \mathrm{Tr}\ \partial\hat{\rho}/\partial t\ \hat{\bar{\Theta}}(\mathbf{r}_1, \mathbf{r}_2) = \frac{i}{\hbar}\mathrm{Tr}\ \hat{\rho}(t)\,[\hat{H}, \hat{\bar{\Theta}}(\mathbf{r}_1, \mathbf{r}_2)] \tag{2.3}$$

where

$$\bar{N}(\mathbf{r}_1, \mathbf{r}_2, t) = \langle\hat{\bar{\Theta}}(\mathbf{r}_1, \mathbf{r}_2)\rangle_t = \mathrm{Tr}\ \hat{\rho}(t)\,\hat{\bar{\Theta}}(\mathbf{r}_1, \mathbf{r}_2) \tag{2.4}$$

and

$$\hat{\bar{\Theta}}(\mathbf{r}_1, \mathbf{r}_2) = \begin{pmatrix} \hat{\Psi}_\uparrow(\mathbf{r}_1)\hat{\Psi}_\uparrow^{+}(\mathbf{r}_2) & \hat{\Psi}_\uparrow(\mathbf{r}_1)\hat{\Psi}_\downarrow(\mathbf{r}_2) \\ \hat{\Psi}_\downarrow^{+}(\mathbf{r}_1)\hat{\Psi}_\uparrow^{+}(\mathbf{r}_2) & \Psi_\downarrow^{+}(\mathbf{r}_1)\hat{\Psi}_\downarrow(\mathbf{r}_2) \end{pmatrix} \tag{2.5}$$

(Note that a bar over a function denotes a 2×2 matrix. A caret denotes an operator.) The field operators $\hat{\Psi}_\sigma(\mathbf{r})$ and $\hat{\Psi}_\sigma^{+}(\mathbf{r})$ refer to the lab frame. We can transform to a frame in which there is no superfluid velocity by performing a gauge transformation. Thus, we write the field operators in the form

$$\hat{\Psi}_\sigma(\mathbf{r}) = \{\exp[i\phi(\mathbf{r})]\}\hat{\psi}_\sigma(\mathbf{r}), \qquad \hat{\Psi}_\sigma^{+}(\mathbf{r}) = \{\exp[-i\phi(\mathbf{r})]\}\hat{\psi}_\sigma^{+}(\mathbf{r}) \tag{2.6}$$

where the field operators $\hat{\psi}_\sigma(\mathbf{r})$ and $\hat{\psi}_\sigma{}^+(\mathbf{r})$ refer to the superfluid rest frame. The superfluid velocity is defined by

$$\mathbf{v}_s(\mathbf{r}) = (\hbar/m)\, \nabla_{\mathbf{r}}\phi \tag{2.7}$$

For simplicity, we assume that the phase is independent of spin. We now can rewrite the Hamiltonian in the form

$$\hat{H} = \hat{H}_0 + \hat{V} \tag{2.8}$$

where

$$\hat{H}_0 = \sum_{\sigma=\uparrow,\downarrow} \int d\mathbf{r}\hat{\psi}_\sigma{}^+(\mathbf{r})\left\{\frac{1}{2m}[-i\hbar\,\nabla_{\mathbf{r}} + m\mathbf{v}_s(\mathbf{r})]^2 - \mu^0\right\}\hat{\psi}_\sigma(\mathbf{r}) + \hat{U} \tag{2.9}$$

and

$$\hat{V} = \iint d\mathbf{r}_1\, d\mathbf{r}_2\, V(|\mathbf{r}_1 - \mathbf{r}_2|)\hat{\psi}_\uparrow{}^+(\mathbf{r}_1)\hat{\psi}_\downarrow{}^+(\mathbf{r}_2)\hat{\psi}_\downarrow(\mathbf{r}_2)\psi_\uparrow(\mathbf{r}_1) - \hat{U} \tag{2.10}$$

The operator $\hat{U}$ includes mean field corrections to the free motion of the particles and is defined by

$$\begin{aligned}\hat{U} = \sum_{\sigma=\uparrow,\downarrow} \int d\mathbf{r}\, F_\sigma(\mathbf{r})\hat{\psi}_\sigma{}^+(\mathbf{r})\hat{\psi}_\sigma(\mathbf{r}) - \iint d\mathbf{r}_1\, d\mathbf{r}_2\, \Delta^+(\mathbf{r}_1, \mathbf{r}_2)\hat{\psi}_\downarrow(\mathbf{r}_1)\hat{\psi}_\uparrow(\mathbf{r}_2) \\ - \iint d\mathbf{r}_1\, d\mathbf{r}_2\, \Delta(\mathbf{r}_1, \mathbf{r}_2)\hat{\psi}_\uparrow{}^+(\mathbf{r}_1)\hat{\psi}_\downarrow{}^+(\mathbf{r}_2)\end{aligned} \tag{2.11}$$

where

$$F_\uparrow(\mathbf{r}) = \int d\mathbf{r}'\, V(|\mathbf{r} - \mathbf{r}'|)\langle\hat{\Psi}_\downarrow{}^+(\mathbf{r}')\hat{\Psi}_\downarrow(\mathbf{r}')\rangle \tag{2.12}$$

$$F_\downarrow(\mathbf{r}) = -\int d\mathbf{r}'\, V(|\mathbf{r} - \mathbf{r}'|)\langle\hat{\Psi}_\uparrow(\mathbf{r}')\Psi_\uparrow{}^+(\mathbf{r}')\rangle \tag{2.13}$$

$$\Delta(\mathbf{r}_1, \mathbf{r}_2) = V(|\mathbf{r}_1 - \mathbf{r}_2|)\langle\hat{\Psi}_\uparrow(\mathbf{r}_1)\hat{\Psi}_\downarrow(\mathbf{r}_2)\rangle \tag{2.14}$$

$$\Delta^+(\mathbf{r}_1, \mathbf{r}_2) = V(|\mathbf{r}_1 - \mathbf{r}_2|)\langle\hat{\Psi}_\downarrow{}^+(\mathbf{r}_1)\hat{\Psi}_\uparrow{}^+(\mathbf{r}_2)\rangle \tag{2.15}$$

The mean field operator $\hat{U}$ contains all possible ways of averaging the field operators in the interaction term in Eq. (2.1) in pairs. The functions $F_\sigma(\mathbf{r})$ describe the change in the energy of particles due to their interaction with the background medium. The functions $\Delta(\mathbf{r}_1, \mathbf{r}_2)$ and $\Delta^+(\mathbf{r}_1, \mathbf{r}_2)$ are only nonzero if the gauge symmetry of the state describing the system is broken. That is, if the Fermi system is in a superfluid state.

Before deriving the kinetic equation it is useful to introduce the operator

$\hat{\hat{\theta}}(\mathbf{r}_1, \mathbf{r}_2)$, whose expectation value is the one-body reduced density matrix in the superfluid rest frame

$$\hat{\hat{\theta}}(\mathbf{r}_1, \mathbf{r}_2) = \begin{pmatrix} \hat{\psi}_\uparrow(\mathbf{r}_1)\hat{\psi}_\uparrow{}^+(\mathbf{r}_2) & \hat{\psi}_\uparrow(\mathbf{r}_1)\hat{\psi}_\downarrow(\mathbf{r}_2) \\ \hat{\psi}_\downarrow{}^+(\mathbf{r}_1)\hat{\psi}_\downarrow{}^+(\mathbf{r}_2) & \hat{\psi}_\downarrow{}^+(\mathbf{r}_1)\hat{\psi}_\downarrow(\mathbf{r}_2) \end{pmatrix} \tag{2.16}$$

It is also useful to write the equation of motion of this operator under the action of the free Hamiltonian $\hat{H}_0$. If we start with the equation

$$i\hbar\, \partial\hat{\hat{\Theta}}(\mathbf{r}_1, \mathbf{r}_2)/\partial t = -[\hat{H}_0, \hat{\hat{\Theta}}(\mathbf{r}_1, \mathbf{r}_2)] \tag{2.17}$$

we can separate out the phase and obtain the following equations for elements of the matrix $\hat{\hat{\theta}}(\mathbf{r}_1, \mathbf{r}_2)$:

$$i\hbar \frac{\partial \hat{\theta}_{11}(\mathbf{r}_1, \mathbf{r}_2)}{\partial t} = \epsilon_\uparrow{}^+(\mathbf{r}_1)\hat{\theta}_{11}(\mathbf{r}_1, \mathbf{r}_2) - \epsilon_\uparrow{}^-(\mathbf{r}_2)\hat{\theta}_{11}(\mathbf{r}_1, \mathbf{r}_2) - \int d\mathbf{r}\, [\Delta(\mathbf{r}_1, \mathbf{r})\hat{\theta}_{21}(\mathbf{r}, \mathbf{r}_2) - \hat{\theta}_{12}(\mathbf{r}_1, \mathbf{r})\Delta^+(\mathbf{r}, \mathbf{r}_2)] \tag{2.17a}$$

$$i\hbar \frac{\partial \hat{\theta}_{21}(\mathbf{r}_1, \mathbf{r}_2)}{\partial t} = \epsilon_\uparrow{}^+(\mathbf{r}_1)\hat{\theta}_{12}(\mathbf{r}_1, \mathbf{r}_2) + \epsilon_\downarrow{}^+(\mathbf{r}_2)\hat{\theta}_{12}(\mathbf{r}_1, \mathbf{r}_2) - \int d\mathbf{r}\, [\Delta(\mathbf{r}_1, \mathbf{r})\hat{\theta}_{22}(\mathbf{r}_1, \mathbf{r}_2) - \hat{\theta}_{11}(\mathbf{r}_1, \mathbf{r})\Delta(\mathbf{r}, \mathbf{r}_2)] \tag{2.17b}$$

$$i\hbar \frac{\partial \hat{\theta}_{21}(\mathbf{r}_1, \mathbf{r}_2)}{\partial t} = -\epsilon_\downarrow{}^-(\mathbf{r}_1)\hat{\theta}_{21}(\mathbf{r}_1, \mathbf{r}_2) - \epsilon_\uparrow{}^-(\mathbf{r}_2)\hat{\theta}_{21}(\mathbf{r}_1, \mathbf{r}_2) - \int d\mathbf{r}\, [\Delta^+(\mathbf{r}_1, \mathbf{r})\hat{\theta}_{11}(\mathbf{r}, \mathbf{r}_2) - \hat{\theta}_{22}(\mathbf{r}_1, \mathbf{r})\Delta^+(\mathbf{r}, \mathbf{r}_2)] \tag{2.17c}$$

$$i\hbar \frac{\partial \hat{\theta}_{22}(\mathbf{r}_1, \mathbf{r}_2)}{\partial t} = -\epsilon_\downarrow{}^-(\mathbf{r}_1)\hat{\theta}_{22}(\mathbf{r}_1, \mathbf{r}_2) + \epsilon_\downarrow{}^+(\mathbf{r}_2)\hat{\theta}_{22}(\mathbf{r}_1, \mathbf{r}_2) - \int d\mathbf{r}\, [\Delta^+(\mathbf{r}_1, \mathbf{r})\hat{\theta}_{12}(\mathbf{r}, \mathbf{r}_2) - \hat{\theta}_{21}(\mathbf{r}_1, \mathbf{r})\Delta(\mathbf{r}, \mathbf{r}_2)] \tag{2.17d}$$

where

$$\epsilon_\sigma{}^\pm(\mathbf{r}) = \frac{1}{2m}[-i\hbar\, \nabla_\mathbf{r} \pm m\mathbf{v}_s(\mathbf{r})]^2 + F_\sigma(\mathbf{r}) + \hbar \frac{\partial \phi}{\partial t} - \mu^0 \tag{2.18}$$

The operator $\hat{\theta}_{ij}$ is the ijth element of the operator matrix $\hat{\hat{\theta}}$. Now that we have obtained the equations of motion for $\hat{\hat{\theta}}(\mathbf{r}_1, \mathbf{r}_2)$, it is convenient to change to the momentum representation. We can change from field operators to particle creation and annihilation operators in the usual manner

$$\hat{\psi}_\sigma(\mathbf{r}) = \frac{1}{V}\sum_{\mathbf{k}} [\exp(i\mathbf{k}\cdot\mathbf{r})]\hat{a}_{\mathbf{k},\sigma}, \qquad \hat{\psi}_\sigma{}^+(\mathbf{r}) = \frac{1}{V}\sum_{\mathbf{k}} [\exp(-i\mathbf{k}\cdot\mathbf{r})]\hat{a}^+_{\mathbf{k},\sigma} \tag{2.19}$$

If we use the convention

$$\hat{\hat{\theta}}(\mathbf{k}_1, \mathbf{k}_2) = \iint d\mathbf{r}_1\, d\mathbf{r}_2\, [\exp(-i\mathbf{k}_1\cdot\mathbf{r}_1)\exp(i\mathbf{k}_2\cdot\mathbf{r}_2)]\hat{\hat{\theta}}(\mathbf{r}_1, \mathbf{r}_2) \tag{2.20}$$

we obtain the following matrix:

$$\hat{\hat{\theta}}(\mathbf{k}_1, \mathbf{k}_2) = \begin{pmatrix} \hat{a}_{\mathbf{k}_1,\uparrow}\hat{a}^+_{\mathbf{k}_2,\uparrow} & \hat{a}_{\mathbf{k}_1,\uparrow}\hat{a}_{-\mathbf{k}_2,\downarrow} \\ \hat{a}^+_{-\mathbf{k}_1,\downarrow}\hat{a}^+_{\mathbf{k}_2,\uparrow} & \hat{a}^+_{-\mathbf{k}_1,\downarrow}\hat{a}_{-\mathbf{k}_2,\downarrow} \end{pmatrix} \tag{2.21}$$

We are now ready to write the kinetic equation.

The major assumption in the work of Peletminskii and Yatsenko[3] is that after a long time, the density matrix $\hat{\rho}(t)$ becomes a functional only of the one-body reduced density matrix. That is, it can be written in the form

$$\hat{\rho}(t) = \hat{\rho}\{\hat{\hat{\theta}}, t\} + \hat{\rho}'\{\hat{\rho}(0)\}e^{-t/\tau_0} \tag{2.22}$$

where $\rho\{\hat{\hat{\theta}}, t\}$ is a functional only of the matrix $\hat{\hat{\theta}}$. This idea was originally due to Bogoliubov and used by him to study classical systems. The quantity $\hat{\rho}(0)$ is the density operator at time $t = 0$, and τ_0 is some relaxation time. After a time much longer than τ_0, we can describe the state of the system by $\hat{\rho}\{\hat{\hat{\theta}}, t\}$. For a superfluid system, $\hat{\rho}\{\hat{\hat{\theta}}, t\}$ has broken gauge symmetry. Furthermore, because $\hat{\rho}\{\hat{\hat{\theta}}, t\}$ depends only on $\hat{\hat{\theta}}$ and therefore has the form of a generalized Gaussian, we can use Wick's theorem to evaluate the expectation value of products of creation and annihilation operators. The actual derivation of the kinetic equation is somewhat involved but straightforward. The method is discussed in Refs. 4 and 5, and we will not repeat the calculations here. The kinetic equation obtained by this method takes the form

$$i\hbar\frac{\partial\bar{n}(\mathbf{k}_1, \mathbf{k}_2, t)}{\partial t} = \frac{1}{V}\sum_{\mathbf{k}_a}\Bigg[\bar{\epsilon}(\mathbf{k}_1, \mathbf{k}_a)\bar{n}(\mathbf{k}_a, \mathbf{k}_2, t) - \bar{n}(\mathbf{k}_1, \mathbf{k}_a, t)\bar{\epsilon}(\mathbf{k}_a, \mathbf{k}_2)$$
$$-\frac{i}{\hbar}\lim_{\delta\to 0}\int_{-\infty}^{0} ds\, e^{\delta s}\,\mathrm{Tr}\{\hat{\rho}\{\hat{\hat{\theta}}, t\}[\hat{V}'(s), [\hat{V}', \hat{\hat{\theta}}(\mathbf{k}_1, \mathbf{k}_2)]]\}\Bigg] \tag{2.23}$$

where

$$\bar{n}(\mathbf{k}_1, \mathbf{k}_2, t) = \langle \hat{\hat{\theta}}(\mathbf{k}_1, \mathbf{k}_2) \rangle_t \tag{2.24}$$

$$\bar{\epsilon}(\mathbf{k}_1, \mathbf{k}_2) = \begin{pmatrix} \epsilon_{\uparrow}^{(-)}(\mathbf{k}_1, \mathbf{k}_2) & -\Delta(\mathbf{k}_1, \mathbf{k}_2) \\ -\Delta^{+}(\mathbf{k}_1, \mathbf{k}_2) & -\epsilon_{\downarrow}^{(+)}(\mathbf{k}_1, \mathbf{k}_2) \end{pmatrix} \tag{2.25}$$

and

$$\epsilon_{\sigma}^{(\pm)}(\mathbf{k}_1, \mathbf{k}_2) = \frac{1}{2m} [\hbar^2 k_1{}^2 \, \delta_{\mathbf{k}_1, \mathbf{k}_2} \pm m\hbar(\mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{v}_s(\mathbf{k}_1 - \mathbf{k}_2) + m v_s{}^2(\mathbf{k}_1 - \mathbf{k}_2)]$$

$$+ F_{\sigma}(\mathbf{k}_1 - \mathbf{k}_2) + \hbar \frac{\partial \phi(\mathbf{k}_1 - \mathbf{k}_2)}{\partial t} \tag{2.26}$$

The Fourier transform of the superfluid velocity $v_s(\mathbf{k}_1 - \mathbf{k}_2)$ is defined by

$$\mathbf{v}_s(\mathbf{k}_1 - \mathbf{k}_2) = \int d\mathbf{r} \, \{\exp[-i(\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{r}]\} \mathbf{v}_s(\mathbf{r}) \tag{2.27}$$

and a similar convention is used for other quantities in Eq. (2.26). The interaction operator $\hat{V}'$ in Eq. (3.23) does not include the mean field terms. Thus,

$$\hat{V}' = \frac{1}{V} \sum_{\mathbf{k}_1, \ldots, \mathbf{k}_4} V(\mathbf{k}_1, \ldots, \mathbf{k}_4) \hat{a}_{\mathbf{k}_1, \uparrow}^{+} \hat{a}_{-\mathbf{k}_2, \downarrow}^{+} \hat{a}_{-\mathbf{k}_4, \downarrow} \hat{a}_{\mathbf{k}_3, \uparrow} \tag{2.28}$$

where

$$V(\mathbf{k}_1, \ldots, \mathbf{k}_4) = \delta(\mathbf{k}_1 - \mathbf{k}_2 + \mathbf{k}_4 - \mathbf{k}_3) I(\mathbf{k}_1 - \mathbf{k}_3) \tag{2.29}$$

and

$$I(\mathbf{k}_1 - \mathbf{k}_3) = \int d\mathbf{r} \, \{\exp[-i\mathbf{r} \cdot (\mathbf{k}_1 - \mathbf{k}_3)]\} \, V(|\mathbf{r}|) \tag{2.30}$$

The first terms on the right-hand side of Eq. (2.23) are streaming terms and will be recognized as the Fourier transforms of Eqs. (2.17a)–(2.17d). The last term on the right-hand side is the collision integral. We notice that the collision integral now depends only on $\hat{V}'$ and does not contain contributions from the mean field terms. These cancel identically when the trace is taken, so we have not included them in Eq. (2.23). The operator $\hat{V}'(s)$ which appears in the collision integral is defined by

$$\hat{V}'(s) = \exp(i\hat{\mathscr{H}}_0 s/\hbar) \, \hat{V}' \exp(-i\hat{\mathscr{H}}_0 s/\hbar) \tag{2.31}$$

where

$$\hat{\mathscr{H}}_0 = \sum_{\mathbf{k}_1, \mathbf{k}_2} \hat{A}_{\mathbf{k}_1}^{+} \bar{\epsilon}(\mathbf{k}_1, \mathbf{k}_2) \hat{A}_{\mathbf{k}_2} \tag{2.32}$$

and $\hat{A}_{\mathbf{k}_1}^+$ and $\hat{A}_{\mathbf{k}_2}$ are column and row vectors and are defined by

$$\hat{A}_{\mathbf{k}}^+ = \widehat{\underbrace{\hat{a}_{\mathbf{k},\uparrow}^+, \hat{a}_{-\mathbf{k},\downarrow}}} \qquad \text{and} \qquad \hat{A}_{\mathbf{k}} = \begin{pmatrix} \hat{a}_{\mathbf{k},\uparrow} \\ \hat{a}_{-\mathbf{k},\downarrow}^+ \end{pmatrix} \tag{2.33}$$

Equation (2.23) is the kinetic equation of a Fermi superfluid in which only antiparallel spin particles interact. In order to make contact with a classical kinetic equation, we can rewrite $\bar{n}(\mathbf{k}_1, \mathbf{k}_2, t)$ in terms of center-of-mass momentum $\mathbf{K} = \frac{1}{2}(\mathbf{k}_1 + \mathbf{k}_2)$ and relative momentum $\mathbf{q} = \mathbf{k}_1 - \mathbf{k}_2$. If we now Fourier transform the $\mathbf{q}$ dependence of $\bar{n}(\mathbf{k}_1, \mathbf{k}_2, t)$, we find

$$\bar{n}(\mathbf{K}, \mathbf{R}, t) = \frac{1}{V} \sum_{\mathbf{q}} [\exp(i\mathbf{q}\cdot\mathbf{R})] \bar{n}\left(\mathbf{K} + \frac{\mathbf{q}}{2}, \mathbf{K} - \frac{\mathbf{q}}{2}\right) \tag{2.34}$$

$\bar{n}(\mathbf{K}, \mathbf{R}, t)$ is the Wigner function for the system. In the classical limit the diagonal elements become proportional to the single-particle distribution function, and the diagonal elements in Eq. (2.23) reduce to the Boltzmann equation.

3. LINEARIZED KINETIC EQUATION

We will now linearize the kinetic equation following closely an analysis by Betbeder-Matibet and Nozières.[6] This method provides a means of closing the kinetic equation when superfluid flow is present. In order to keep our discussion as simple as possible, we will specialize our Hamiltonian to the Gor'kov[7] model. That is, we let

$$V(|\mathbf{r}_1 - \mathbf{r}_2|) = g\, \delta(\mathbf{r}_1 - \mathbf{r}_2) \tag{3.1}$$

where g is a small, negative coupling constant which is only nonzero in a small interval about the Fermi surface. With this interaction the functions appearing in Eq. (2.25) take the form

$$F_\uparrow(\mathbf{q}) = \frac{g}{V} \sum_{\mathbf{K}} n_{22}(\mathbf{K}, \mathbf{q}) \tag{3.2}$$

$$F_\downarrow(\mathbf{q}) = -\frac{g}{V} \sum_{\mathbf{K}} n_{11}(\mathbf{K}, \mathbf{q}) \tag{3.3}$$

$$\Delta(\mathbf{k}_1, \mathbf{k}_2) = \Delta(\mathbf{q}) = \frac{g}{V} \sum_{\mathbf{K}} n_{12}(\mathbf{K}, \mathbf{q}) \tag{3.4}$$

$$\Delta^+(\mathbf{k}_1, \mathbf{k}_2) = \Delta^+(\mathbf{q}) = \frac{g}{V} \sum_{\mathbf{K}} n_{21}(\mathbf{K}, \mathbf{q}) \tag{3.5}$$

$$I(\mathbf{k}_1 - \mathbf{k}_3) = g \tag{3.6}$$

We have used the notation $n_{ij}(\mathbf{K}+\frac{1}{2}\mathbf{q}, \mathbf{K}-\frac{1}{2}\mathbf{q}) = n_{ij}(\mathbf{K}, \mathbf{q})$. The quantities $\Delta(\mathbf{q})$ and $\Delta^+(\mathbf{q})$ are the nonequilibrium gap functions. They vary in position but are independent of momentum.

Let us now linearize the kinetic equation about absolute equilibrium. We will write

$$\bar{\epsilon}(\mathbf{k}_1, \mathbf{k}_2) = \bar{\epsilon}_0(\mathbf{k}_1)\, \delta_{\mathbf{k}_1,\mathbf{k}_2} + \delta\bar{\epsilon}(\mathbf{k}_1, \mathbf{k}_2) \tag{3.7}$$

and

$$\bar{n}(\mathbf{k}_1, \mathbf{k}_2) = \bar{n}_0(\mathbf{k}_1)\, \delta_{\mathbf{k}_1,\mathbf{k}_2} + \delta\bar{n}(\mathbf{k}_1, \mathbf{k}_2) \tag{3.8}$$

where

$$\bar{\epsilon}_0(\mathbf{k}) = \begin{pmatrix} \xi_{\mathbf{k}} & -\Delta \\ -\Delta^* & -\xi_{\mathbf{k}} \end{pmatrix} \tag{3.9}$$

is the equilibrium energy matrix, $\xi_{\mathbf{k}} = \hbar^2 k^2/2m - \mu$, $\mu = \mu^0 + \frac{1}{2}n$, and $\Delta^* = -\Delta$. We have assumed that the equilibrium densities of spin-$\uparrow$ and spin-$\downarrow$ particles are equal, i.e., $n_\uparrow = n_\downarrow = n$. We also have assumed that at equilibrium there is no superfluid velocity (a constant finite superfluid velocity is also an equilibrium state). The equilibrium particle distribution function $\bar{n}_0(\mathbf{k}_1)$ is defined by

$$\bar{n}_0(\mathbf{k}_1) = \tfrac{1}{2}[\bar{I} + \bar{\epsilon}_0(\mathbf{k}_1)(E^0_{\mathbf{k}_1})^{-1} \tanh(\tfrac{1}{2}\beta E^0_{\mathbf{k}_1})] \tag{3.10}$$

where $\bar{I}$ is the 2×2 unit matrix, and

$$E_{\mathbf{k}}{}^0 = (\xi_{\mathbf{k}}{}^2 + |\Delta|^2)^{1/2} \tag{3.11}$$

is the equilibrium bogolon energy. The linearized kinetic equation now takes the form

$$\begin{aligned} i\hbar\, \partial\delta\bar{n}(\mathbf{k}_1, \mathbf{k}_2)/\partial t &\\ = \bar{\epsilon}_0(\mathbf{k}_1)\, \delta\bar{n}(\mathbf{k}_1, \mathbf{k}_2) &- \delta\bar{n}(\mathbf{k}_1, \mathbf{k}_2)\bar{\epsilon}_0(\mathbf{k}_2) \\ + \delta\bar{\epsilon}(\mathbf{k}_1, \mathbf{k}_2)\bar{n}_0(\mathbf{k}_2) &- \bar{n}_0(\mathbf{k}_1)\, \delta\bar{\epsilon}(\mathbf{k}_1, \mathbf{k}_2) + \mathbb{C}_{\text{lin}}(\hat{\bar{\theta}}(\mathbf{k}_1, \mathbf{k}_2)) \end{aligned} \tag{3.12}$$

where $\mathbb{C}_{\text{lin}}(\hat{\theta}(\mathbf{k}_1, \mathbf{k}_2))$ denotes the linearized collision integral.

The properties of a Fermi superfluid are best described in terms of the collective modes, called bogolons, rather than particles. We can transform to a description in terms of bogolons via a unitary transformation (the Bogoliubov transformation[8]) which diagonalizes the energy matrix in Eq. (3.9). The resulting bogolon kinetic equation can be written as a set of scalar equations, and the collision integral contains information about collisions between bogolons rather than particles. Let us first introduce the following transformation from particle operators $\hat{a}_{\mathbf{k},\sigma}$ and $\hat{a}^+_{\mathbf{k},\sigma}$ to bogolon operators $\hat{b}_{\mathbf{k},\lambda}$ and $\hat{b}^+_{\mathbf{k},\lambda}$. We write

$$\hat{B}_{\mathbf{k}} = \begin{pmatrix} \hat{b}_{\mathbf{k},+} \\ \hat{b}^+_{-\mathbf{k},-} \end{pmatrix} = U_{\mathbf{k}}{}^+ \begin{pmatrix} \hat{a}_{\mathbf{k},\uparrow} \\ \hat{a}^+_{-\mathbf{k},\downarrow} \end{pmatrix} \tag{3.13}$$

where

$$\bar{U}_{\mathbf{k}}{}^+ = \begin{pmatrix} u_{\mathbf{k}} & -v_{\mathbf{k}} \\ v_{\mathbf{k}}{}^* & u_{\mathbf{k}}{}^* \end{pmatrix} \tag{3.14}$$

(the asterisk indicates complex conjugation) and the elements of the transformation matrix are defined by

$$u_{\mathbf{k}} v_{\mathbf{k}} = \tfrac{1}{2}\Delta/E_{\mathbf{k}}{}^0, \qquad u_{\mathbf{k}} v_{\mathbf{k}}{}^* = \tfrac{1}{2}\Delta^*/E_{\mathbf{k}}{}^0 \tag{3.15a}$$

$$u_{\mathbf{k}} = [\tfrac{1}{2}(1 + \xi_{\mathbf{k}}/E_{\mathbf{k}}{}^0)]^{1/2} \tag{3.15b}$$

and

$$|v_{\mathbf{k}}| = [\tfrac{1}{2}(1 - \xi_{\mathbf{k}}/E_{\mathbf{k}}{}^0)]^{1/2} \tag{3.15c}$$

As usual, we have assumed that $u_{\mathbf{k}}$ is real. The transformation matrix is unitary:

$$\bar{U}_{\mathbf{k}}{}^+ \bar{U}_{\mathbf{k}} = \bar{U}_{\mathbf{k}} \bar{U}_{\mathbf{k}}{}^+ = \bar{I} \tag{3.16}$$

This transformation diagonalizes the energy matrix

$$\bar{E}^0(\mathbf{k}) = \begin{pmatrix} E_{\mathbf{k}}{}^0 & 0 \\ 0 & -E_{\mathbf{k}}{}^0 \end{pmatrix} = \bar{U}_{\mathbf{k}}{}^+ \bar{\epsilon}_{\mathbf{k}}{}^0 \bar{U}_{\mathbf{k}} \tag{3.17}$$

and enables us to transform to the one-body reduced density matrix for bogolons $\langle \hat{\bar{\gamma}}(\mathbf{k}_1, \mathbf{k}_2) \rangle_t$, where

$$\hat{\bar{\gamma}}(\mathbf{k}_1, \mathbf{k}_2) = \bar{U}^+_{\mathbf{k}_1} \hat{\bar{\theta}}(\mathbf{k}_1, \mathbf{k}_2) \bar{U}_{\mathbf{k}_2} = \begin{pmatrix} \hat{b}_{\mathbf{k}_1,+} \hat{b}^+_{\mathbf{k}_2,+} & \hat{b}_{\mathbf{k}_1,+} \hat{b}_{-\mathbf{k}_2,-} \\ \hat{b}^+_{-\mathbf{k}_1,-} \hat{b}^+_{\mathbf{k}_2,+} & \hat{b}^+_{-\mathbf{k}_1,-} \hat{b}_{-\mathbf{k}_2,-} \end{pmatrix} \tag{3.18}$$

The particle distribution matrix $\bar{n}_{\mathbf{k}}{}^0$ transforms to the bogolon distribution matrix $\bar{\nu}_{\mathbf{k}}{}^0$,

$$\bar{\nu}_{\mathbf{k}}{}^0 = \bar{U}_{\mathbf{k}}{}^+ \bar{n}_{\mathbf{k}}{}^0 \bar{U}_{\mathbf{k}} = \tfrac{1}{2}[\bar{I} + \bar{\tau}_3 \tanh(\beta E_{\mathbf{k}}{}^0/2)] \tag{3.19}$$

where

$$\bar{\tau}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The deviation from equilibrium of the bogolon distribution is given by

$$\delta\bar{v}(\mathbf{k}_1, \mathbf{k}_2, t) = \bar{U}^{+}_{\mathbf{k}_1}\, \delta\bar{n}(\mathbf{k}_1, \mathbf{k}_2, t)\bar{U}_{\mathbf{k}_2} \tag{3.20}$$

and the energy corrections are given by

$$\delta\bar{E}(\mathbf{k}_1, \mathbf{k}_2, t) = \bar{U}^{+}_{\mathbf{k}_1}\, \delta\bar{\varepsilon}(\mathbf{k}_1, \mathbf{k}_2)\bar{U}_{\mathbf{k}_2} \tag{3.21}$$

We now can transform the kinetic equation (3.11) to a bogolon kinetic equation

$$\begin{aligned} i\hbar\, &\partial\delta\bar{v}(\mathbf{k}_1, \mathbf{k}_2, t)/\partial t \\ &= \bar{E}_0(\mathbf{k}_1)\, \delta\bar{v}(\mathbf{k}_1, \mathbf{k}_2, t) - \delta\bar{v}(\mathbf{k}_1, \mathbf{k}_2, t)E_0(\mathbf{k}_2) \\ &\quad + \delta E(\mathbf{k}_1, \mathbf{k}_2, t)\bar{v}_0(\mathbf{k}_2) - \bar{v}_0(\mathbf{k}_1)\, \delta\bar{E}(\mathbf{k}_1, \mathbf{k}_2, t) + \mathbb{C}_{\text{lin}}(\hat{\bar{\gamma}}(\mathbf{k}_1, \mathbf{k}_2)) \end{aligned} \tag{3.22}$$

In order to make contact with physical quantities, we transform to center-of-mass momentum $\mathbf{K} = \frac{1}{2}(\mathbf{k}_1 + \mathbf{k}_2)$ and relative momentum $\mathbf{q} = \mathbf{k}_1 - \mathbf{k}_2$. As we saw at the end of Section 2, $\mathbf{q}$ is the wave vector for spatial disturbances.

Since we are interested in obtaining the hydrodynamic modes, we only need to retain contributions for small $\mathbf{q}$ (long wavelength). Thus we expand the streaming terms in powers of $\mathbf{q}$ and retain lowest order terms. The kinetic equation then takes the form

$$\begin{aligned} i\hbar\, &\partial\delta\bar{v}(\mathbf{K}, \mathbf{q}, t)/\partial t \\ &= E_{\mathbf{K}}{}^0[\bar{\tau}_3\, \delta\bar{v}(\mathbf{K}, \mathbf{q}, t) - \delta\bar{v}(\mathbf{K}, \mathbf{q}, t)\bar{\tau}_3] \\ &\quad + \theta_{\mathbf{k}}{}^0[\delta\bar{E}(\mathbf{K}, \mathbf{q}, t)\bar{\tau}_3 - \bar{\tau}_3\, \delta\bar{E}(\mathbf{K}, \mathbf{q}, t)] \\ &\quad + \frac{1}{2}\frac{\hbar^2}{m}\mathbf{K}\cdot\mathbf{q}\,\frac{\xi_{\mathbf{K}}}{E_{\mathbf{K}}{}^0}\,[\bar{\tau}_3\, \delta\bar{v}(\mathbf{K}, \mathbf{q}, t) - \delta\bar{v}(\mathbf{K}, \mathbf{q}, t)\bar{\tau}_3] \\ &\quad - \frac{1}{2}\frac{\hbar^2}{m}\mathbf{K}\cdot\mathbf{q}\,\frac{\xi_{\mathbf{K}}}{E_{\mathbf{K}}{}^0}\,(\partial\theta_{\mathbf{K}}{}^0/\partial E_{\mathbf{K}}{}^0)[\delta\bar{E}(\mathbf{K}, \mathbf{q}, t)\bar{\tau}_3 + \bar{\tau}_3\, \delta\bar{E}(\mathbf{K}, \mathbf{q}, t)] \\ &\quad + \mathbb{C}_{\text{lin}}(\hat{\bar{\gamma}}(\mathbf{K} + \tfrac{1}{2}\mathbf{q}, \mathbf{K} - \tfrac{1}{2}\mathbf{q})) \end{aligned} \tag{3.23}$$

where

$$\theta_{\mathbf{K}}{}^0 = +\tfrac{1}{2}\tanh(\beta E_{\mathbf{K}}{}^0/2) \tag{3.24}$$

It is useful to write the various elements of the matrix equation (3.22) explicitly:

$$i\hbar\,\frac{\partial\delta\nu_{11}(\mathbf{K},\mathbf{q},t)}{\partial t}-\frac{\hbar^2}{m}\,\mathbf{K}\cdot\mathbf{q}\,\frac{\xi_{\mathbf{K}}}{E_{\mathbf{K}}{}^0}\,\delta\nu_{11}(\mathbf{K},\mathbf{q},t)$$

$$+\frac{\hbar^2}{m}\,\mathbf{K}\cdot\mathbf{q}\,\frac{\xi_{\mathbf{K}}}{E_{\mathbf{K}}{}^0}\,\frac{\partial\theta_{\mathbf{K}}{}^0}{\partial E_{\mathbf{K}}{}^0}\,\delta E_{11}(\mathbf{K},\mathbf{q},t)=\mathbb{C}_{\text{lin}}(\hat{b}_{\mathbf{K}+\mathbf{q}/2,+}\hat{b}^{+}_{\mathbf{K}-\mathbf{q}/2,+}) \tag{3.25a}$$

$$i\hbar\,\frac{\partial\delta\nu_{22}(\mathbf{K},\mathbf{q},t)}{\partial t}+\frac{\hbar^2}{m}\,\mathbf{K}\cdot\mathbf{q}\,\frac{\xi_{\mathbf{K}}}{E_{\mathbf{K}}{}^0}\,\delta\nu_{22}(\mathbf{K},\mathbf{q},t)$$

$$-\frac{\hbar^2}{m}\,\mathbf{K}\cdot\mathbf{q}\,\frac{\xi_{\mathbf{K}}}{E_{\mathbf{K}}{}^0}\,\frac{\partial\theta_{\mathbf{K}}{}^0}{\partial E_{\mathbf{k}}{}^0}\,\delta E_{22}(\mathbf{K},\mathbf{q},t)=\mathbb{C}_{\text{lin}}(\hat{b}^{+}_{-\mathbf{K}-\mathbf{q}/2,-}\hat{b}_{-\mathbf{K}+\mathbf{q}/2,-}) \tag{3.25b}$$

$$i\hbar\,\frac{\partial\delta\nu_{12}(\mathbf{K},\mathbf{q},t)}{\partial t}-2E_{\mathbf{K}}{}^0\,\delta\nu_{12}(\mathbf{K},\mathbf{q},t)+2\theta_{\mathbf{K}}{}^0\,\delta E_{12}(\mathbf{K},\mathbf{q})$$

$$=\mathbb{C}_{\text{lin}}(\hat{b}_{\mathbf{K}+\mathbf{q}/2,+}\hat{b}_{-\mathbf{K}+\mathbf{q}/2,-}) \tag{3.25c}$$

$$i\hbar\,\frac{\partial\delta\nu_{21}(\mathbf{K},\mathbf{q},t)}{\partial t}+2E_{\mathbf{K}}{}^0\,\delta\nu_{21}(\mathbf{K},\mathbf{q},t)-2\theta_{\mathbf{K}}{}^0\,\delta E_{21}(\mathbf{K},\mathbf{q},t)$$

$$=\mathbb{C}_{\text{lin}}(\hat{b}^{+}_{-\mathbf{K}-\mathbf{q}/2,-}\hat{b}^{+}_{\mathbf{K}-\mathbf{q}/2,+}) \tag{3.25d}$$

The quantity $(\hbar\mathbf{K}/m)\xi_{\mathbf{K}}/E_{\mathbf{K}}{}^0$ which appears in Eqs. (3.25a) and (3.25b) is the bogolon velocity. We can now begin the process of obtaining closed equations for the bogolon distribution functions $\delta\nu_{22}(\mathbf{K},\mathbf{q},t)$ and $\delta\nu_{11}(\mathbf{K},\mathbf{q},t)$.

Let us first note from Eqs. (3.25c) and (3.25d) that for slowly varying processes we can write

$$\delta\nu_{\binom{12}{21}}(\mathbf{K},\mathbf{q},t)=(\theta_{\mathbf{K}}{}^0/E_{\mathbf{K}}{}^0)\,\delta E_{\binom{12}{21}}(\mathbf{K},\mathbf{q},t)+\text{small corrections} \tag{3.26}$$

Let us next look more closely at self-consistent field corrections. From Eqs. (2.25), (3.7), and (3.9), we can write

$$\delta\bar{\epsilon}(\mathbf{K},\mathbf{q},t)=\begin{pmatrix}-\hbar\mathbf{K}\cdot\mathbf{v}_s(\mathbf{q},t)+F_{\uparrow}(\mathbf{q},t)+\hbar\,\partial\phi(\mathbf{q},t)/\partial t & -\delta\Delta(\mathbf{q},t)\\ -\delta\Delta^{+}(\mathbf{q},t) & -\hbar\mathbf{K}\cdot\mathbf{v}_s(\mathbf{q},t)-F_{\downarrow}(\mathbf{q},t)-\hbar\,\partial\phi(\mathbf{q},t)/\partial t\end{pmatrix} \tag{3.27}$$

Then, using the properties of the Bogoliubov transformation, $\hat{U}_{\mathbf{K}}{}^{+}$, and Eqs. (3.2) and (3.3), it is not difficult to show that

$$\begin{aligned}&\delta E_{11}(\mathbf{K},\mathbf{q},t)+\delta E_{22}(\mathbf{K},\mathbf{q},t)\\&\quad=\frac{g}{V}\sum_{K'}\left[\delta n_{22}(-\mathbf{K}',\mathbf{q},t)+\delta n_{11}(\mathbf{K}',\mathbf{q},t)\right]\\&\quad=\frac{g}{V}\sum_{K'}\left[\delta\nu_{22}(-\mathbf{K}',\mathbf{q},t)+\delta\nu_{11}(\mathbf{K}',\mathbf{q},t)\right]\end{aligned} \tag{3.28}$$

Similarly

$$\begin{aligned}
&\delta E_{22}(-\mathbf{K}, \mathbf{q}, t) - \delta E_{11}(\mathbf{K}, \mathbf{q}, t) \\
&\quad = 2\hbar\mathbf{K}\cdot\mathbf{v}_s(\mathbf{q}, t) - 2\hbar \frac{\xi_{\mathbf{K}}}{E_{\mathbf{K}}{}^0} \partial\phi(\mathbf{q}, t)/\partial t \\
&\qquad - \frac{1}{E_{\mathbf{K}}{}^0} [\Delta^* \, \delta\Delta(\mathbf{q}, t) + \Delta \, \delta\Delta^+(\mathbf{q}, t)] \\
&\qquad - \frac{\xi_{\mathbf{K}}}{E_{\mathbf{K}}{}^0} \frac{g}{V} \sum_{\mathbf{K}'} [\delta n_{22}(\mathbf{K}', \mathbf{q}, t) - \delta n_{11}(\mathbf{K}', \mathbf{q}, t)]
\end{aligned} \tag{3.29}$$

(We have neglected terms of order $v_s{}^2$.) We can now write the third and fourth terms on the right-hand side of Eq. (3.29) in terms of bogolon distribution functions. Let us begin with the last term in Eq. (3.28). From Eqs. (3.4) and (3.5) and the Bogoliubov transformation, we can write

$$\begin{aligned}
&\Delta^* \, \delta\Delta(\mathbf{q}, t) + \Delta \, \delta\Delta^+(\mathbf{q}, t) \\
&\quad = \frac{g}{V} \sum_{\mathbf{k}} [\Delta^* \, \delta n_{12}(\mathbf{k}, \mathbf{q}, t) + \Delta \, \delta n_{21}(\mathbf{k}, \mathbf{q}, t)] \\
&\quad = +\frac{g}{V} \sum_{\mathbf{K}} \left\{ \frac{\xi_{\mathbf{K}}}{E_{\mathbf{K}}{}^0} [\Delta^* \, \delta v_{12}(\mathbf{K}, \mathbf{q}, t) + \Delta \, \delta v_{21}(\mathbf{K}, \mathbf{q}, t)] \right. \\
&\qquad \left. + \frac{\Delta_0{}^2}{E_{\mathbf{K}}{}^0} [\delta v_{22}(\mathbf{K}, \mathbf{q}, t) - \delta v_{11}(\mathbf{K}, \mathbf{q}, t)] \right\}
\end{aligned} \tag{3.30}$$

where $\Delta_0{}^2 = \Delta^*\Delta$. But from Eq. (3.26) we find that

$$\begin{aligned}
&\Delta^* \, \delta v_{12}(\mathbf{K}, \mathbf{q}, t) + \Delta \, \delta v_{21}(\mathbf{K}, \mathbf{q}, t) \\
&\quad = \frac{\theta_{\mathbf{K}}{}^0}{E_{\mathbf{K}}{}^0} [\Delta^* \, \delta E_{12}(\mathbf{K}, \mathbf{q}, t) + \Delta \, \delta E_{21}(\mathbf{K}, \mathbf{q}, t)] \\
&\quad = \frac{\theta_{\mathbf{K}}{}^0}{E_{\mathbf{K}}{}^0} \left\{ \frac{\Delta_0{}^2}{E_{\mathbf{K}}{}^0} \left[\delta F_\uparrow(\mathbf{q}, t) + \delta F_\downarrow(\mathbf{q}, t) + 2\hbar \frac{\partial\phi(\mathbf{q}, t)}{\partial t} \right] \right. \\
&\qquad \left. + \frac{\xi_{\mathbf{K}}}{E_{\mathbf{K}}{}^0} [\Delta^* \, \delta\Delta(\mathbf{q}, t) + \Delta \, \delta\Delta^+(\mathbf{q}, t)] \right\}
\end{aligned} \tag{3.31}$$

Combining Eqs. (3.30) and (3.31), we obtain

$$\begin{aligned}
&\Delta^* \,\delta\Delta(\mathbf{q}, t) + \Delta \,\delta\Delta^+(\mathbf{q}, t) \\
&\quad = +\frac{g}{V}\sum_{\mathbf{K}} \frac{\xi_{\mathbf{K}}}{E_{\mathbf{K}}^0}\frac{\theta_{\mathbf{K}}^0}{E_{\mathbf{K}}^0}\left\{\frac{\Delta_0^2}{E_{\mathbf{K}}^0}\left[\delta F_\uparrow(\mathbf{q}, t) + \delta F_\downarrow(\mathbf{q}, t) + 2\hbar\frac{\partial\phi(\mathbf{q}, t)}{\partial t}\right]\right\} \\
&\qquad + \frac{g}{V}\sum_{\mathbf{K}} \frac{\Delta_0^2}{E_{\mathbf{K}}^0}[\delta v_{22}(\mathbf{K}, \mathbf{q}, t) - \delta v_{11}(\mathbf{K}, \mathbf{q}, t)] \\
&\qquad - \frac{g}{V}\sum_{\mathbf{K}} \frac{\theta_{\mathbf{K}}^0}{E_{\mathbf{K}}^0}\left(\frac{\xi_{\mathbf{K}}}{E_{\mathbf{K}}^0}\right)^2[\Delta^* \,\delta\Delta(\mathbf{q}, t) + \Delta\,\delta\Delta^+(\mathbf{q}, t)]
\end{aligned} \tag{3.32}$$

In the term in Eq. (3.32) that involves $\delta F_\uparrow + \delta F_\downarrow + 2\hbar\,\partial\phi/\partial t$ the integration over $\mathbf{K}$ involves an odd function of $\xi_{\mathbf{K}}$ ($\xi_{\mathbf{K}}$ is odd with respect to the Fermi surface) and will be of order Δ/E_F, where E_F is the Fermi energy of the system. Thus, for a weakly coupled system where $\Delta \ll E_F$ that term can be neglected. Then Eq. (3.32) takes the form

$$\begin{aligned}
&\Delta^* \,\delta\Delta(\mathbf{q}, t) + \Delta \,\delta\Delta^+(\mathbf{q}, t) \\
&\quad = +\frac{g}{V}\chi(T)^{-1}\sum_{\mathbf{K}} \frac{\Delta_0^2}{E_{\mathbf{K}}^0}[\delta v_{22}(\mathbf{K}, \mathbf{q}, t) - \delta v_{11}(\mathbf{K}, \mathbf{q}, t)]
\end{aligned} \tag{3.33}$$

where

$$\chi(T) = 1 + \frac{g}{V}\sum_{\mathbf{K}}\left(\frac{\xi_{\mathbf{K}}}{E_{\mathbf{K}}^0}\right)^2\frac{\theta_{\mathbf{K}}^0}{E_{\mathbf{K}}^0} \tag{3.34}$$

Let us now consider the last term in Eq. (3.29). Using Eq. (3.26) and the Bogoliubov transformation, we can write

$$\begin{aligned}
&\delta n_{22}(\mathbf{K}, \mathbf{q}, t) - \delta n_{11}(\mathbf{K}, \mathbf{q}, t) \\
&\quad = \frac{\xi_{\mathbf{K}}}{E_{\mathbf{K}}^0}[\delta v_{22}(\mathbf{K}, \mathbf{q}, t) - \delta v_{11}(\mathbf{K}, \mathbf{q}, t)] \\
&\qquad - \frac{1}{E_{\mathbf{K}}^0}[\Delta^*\,\delta v_{12}(\mathbf{K}, \mathbf{q}, t) + \Delta\,\delta v_{21}(\mathbf{K}, \mathbf{q}, t)] \\
&\quad = \frac{\xi_{\mathbf{K}}}{E_{\mathbf{K}}^0}[\delta v_{22}(\mathbf{K}, \mathbf{q}, t) - \delta v_{11}(\mathbf{K}, \mathbf{q}, t)] \\
&\qquad - \left(\frac{\Delta_0}{E_{\mathbf{K}}^0}\right)^2\frac{\theta_{\mathbf{K}}^0}{E_{\mathbf{K}}^0}\left\{2\hbar\frac{\partial\phi}{\partial t} + \frac{g}{V}\sum_{\mathbf{K}'}[\delta n_{22}(\mathbf{K}', \mathbf{q}, t) - \delta n_{11}(\mathbf{K}', \mathbf{q}, t)]\right\} \\
&\qquad + \frac{1}{E_{\mathbf{K}}^0}\frac{\xi_{\mathbf{K}}}{E_{\mathbf{K}}^0}\frac{\theta_{\mathbf{K}}^0}{E_{\mathbf{K}}^0}[\Delta^*\,\delta\Delta(\mathbf{q}, t) + \Delta\,\delta\Delta^+(\mathbf{q}, t)]
\end{aligned} \tag{3.35}$$

The last term in Eq. (3.35) can be neglected when Eq. (3.35) is substituted into (3.29) because it is odd in $\xi_{\mathbf{K}}$. If we solve Eq. (3.35) for $\delta n_{22} - \delta n_{11}$, we find

$$\begin{aligned}
&\delta n_{22}(\mathbf{K}, \mathbf{q}, t) - \delta n_{11}(\mathbf{K}, \mathbf{q}, t) \\
&\quad = (1 + \hat{F}_{\mathbf{K}})^{-1} \frac{\xi_{\mathbf{K}}}{E_{\mathbf{K}}{}^{0}} [\delta v_{22}(\mathbf{K}, \mathbf{q}, t) - \delta v_{11}(\mathbf{K}, \mathbf{q}, t)] \\
&\qquad - (1 + \hat{F}_{\mathbf{K}})^{-1} \left(\frac{\Delta_0}{E_{\mathbf{K}}{}^{0}}\right)^2 \frac{\theta_{\mathbf{K}}{}^{0}}{E_{\mathbf{K}}{}^{0}} 2\hbar \frac{\partial \phi}{\partial t}
\end{aligned} \tag{3.36}$$

where $\hat{F}_{\mathbf{K}}$ is an integral operator and is defined by

$$\hat{F}_{\mathbf{K}} g(\mathbf{K}) = \frac{g}{V} \left(\frac{\Delta_0}{E_{\mathbf{K}}{}^{0}}\right)^2 \frac{\theta_{\mathbf{K}}{}^{0}}{E_{\mathbf{K}}{}^{0}} \sum_{\mathbf{K}'} g(\mathbf{K}') \tag{3.37}$$

If we now substitute Eqs. (3.36) and (3.33) into Eq. (3.29), we finally obtain

$$\begin{aligned}
&\delta E_{22}(-\mathbf{K}, \mathbf{q}, t) - \delta E_{11}(\mathbf{K}, \mathbf{q}, t) \\
&\quad = 2\hbar \mathbf{K} \cdot \mathbf{v}_s(\mathbf{q}, t) + 2\hbar [1 + \Gamma(T)]^{-1} \frac{\xi_{\mathbf{K}}}{E_{\mathbf{K}}{}^{0}} \frac{\partial \phi(\mathbf{q}, t)}{\partial t} \\
&\qquad - \frac{\xi_{\mathbf{K}}}{E_{\mathbf{K}}{}^{0}} \frac{g}{V} [1 + \Gamma(T)]^{-1} \sum_{\mathbf{K}'} \frac{\xi_{\mathbf{K}'}}{E_{\mathbf{K}'}^{0}} [\delta v_{22}(-\mathbf{K}', \mathbf{q}, t) - \delta v_{11}(\mathbf{K}, \mathbf{q}, t)] \\
&\qquad - \frac{g}{V} \chi(T)^{-1} \frac{\Delta_0{}^2}{E_{\mathbf{K}}{}^{0}} \sum_{\mathbf{K}'} \frac{1}{E_{\mathbf{K}'}^{0}} [\delta v_{22}(-\mathbf{K}', \mathbf{q}, t) - \delta v_{11}(\mathbf{K}', \mathbf{q}, t)]
\end{aligned} \tag{3.38}$$

where

$$\Gamma(T) = \frac{g}{V} \sum_{\mathbf{K}} \left(\frac{\Delta_0}{E_{\mathbf{K}}{}^{0}}\right)^2 \frac{\theta_{\mathbf{K}}{}^{0}}{E_{\mathbf{K}}{}^{0}} \tag{3.39}$$

In the term involving $\delta v_{22}(-\mathbf{K}, \mathbf{q}, t)$, $\mathbf{K}$ is a dummy variable and its sign has been changed by relabeling.

Equations (3.25a), (3.25b), (3.28), and (3.38) almost constitute a closed set of equations for δv_{11} and δv_{22}. The only problem remaining is to express the superfluid velocity and the time derivative of the phase in terms of these quantities. We shall return to this after we have studied the collision integral.

4. COLLISION INTEGRAL

Let us now obtain an explicit expression for the collision integral. We are interested in the following quantity:

$$\mathbb{C}(\hat{b}^{+}_{\lambda\mathbf{K}-\mathbf{q}/2,\lambda}\hat{b}_{\lambda\mathbf{K}+\mathbf{q}/2,\lambda})$$
$$= -\frac{i}{\hbar}\lim_{\delta\to 0}\int_{-\infty}^{0} ds\, e^{\delta s}\,\mathrm{Tr}\{\hat{\rho}\{\hat{\bar{\theta}}, t\}[\hat{V}'(s), [\hat{V}', \hat{b}^{+}_{\lambda\mathbf{K}-\mathbf{q}/2,\lambda}\hat{b}_{\lambda\mathbf{K}+\mathbf{q}/2,\lambda}]]\} \tag{4.1}$$

Let us first consider the Hamiltonian $\hat{\mathscr{H}}_0$ which appears in the expression for $\hat{V}'(s)$,

$$\hat{\mathscr{H}}_0 = \sum_{\mathbf{k}_1,\mathbf{k}_2} \hat{B}^{+}_{\mathbf{k}_1}\bar{E}(\mathbf{k}_1, \mathbf{k}_2)\hat{B}_{\mathbf{k}_2} \tag{4.2}$$

As we can see, this Hamiltonian is not diagonal but contains contributions due to bogolon pairing. It is convenient in deriving the collision integral to assume that spatial disturbances have one wavelength characterized by wave vector $\mathbf{q}$. Let us now retain contributions to lowest order in $\mathbf{q}$ and neglect contributions due to bogolon pairing. Then we can write

$$\hat{\mathscr{H}}_0 \approx \sum_{\lambda}\sum_{\mathbf{K}} E_\lambda(\mathbf{K}, \mathbf{q})\hat{b}^{+}_{\mathbf{K},\lambda}\hat{b}_{\mathbf{K},\lambda} + \text{small corrections} \tag{4.3}$$

where

$$E_{+}(\mathbf{K}, \mathbf{q}) = E_{\mathbf{K}}{}^{0} - \delta E_{11}(\mathbf{K}, \mathbf{q}) \tag{4.4}$$

and

$$E_{-}(\mathbf{K}, \mathbf{q}) = E_{\mathbf{K}}{}^{0} + \delta E_{22}(\mathbf{K}, \mathbf{q}) \tag{4.5}$$

We see that the Hamiltonian involves the bogolon energy in local equilibrium and not absolute equilibrium. This is just what we expect from Fermi liquid theory.

We can now use Wick's theorem to evaluate Eq. (4.1). The calculation is somewhat tedious. The resulting expression will depend on products of pairwise averages $\langle \hat{b}^{+}_{\mathbf{k}',\lambda}\hat{b}_{\mathbf{k},\lambda}\rangle$. We can linearize the collision integral by expanding the pairwise averages about local equilibrium. Thus we write

$$\langle \hat{b}^{+}_{\lambda\mathbf{K}-\mathbf{q}/2,\lambda}\hat{b}_{\lambda\mathbf{K}+\mathbf{q}/2,\lambda}\rangle = \bar{f}^{0}(E_\lambda(\mathbf{K}, \mathbf{q})) + \frac{\partial \bar{f}^{0}(E_\lambda(\mathbf{K}, \mathbf{q}))}{\partial E_\lambda(\mathbf{K}, \mathbf{q})}\Phi_\lambda(\mathbf{K}, \mathbf{q}) \tag{4.6}$$

where $\bar{f}^{0}(E_\lambda(\mathbf{K}, \mathbf{q}))$ is the local equilibrium distribution function

$$\bar{f}^{0}(E_\lambda(\mathbf{K}, \mathbf{q})) = \{\exp[\beta E_\lambda(\mathbf{K}, \mathbf{q})] + 1\}^{-1} = \tfrac{1}{2}\{1 + \tanh[\beta E_\lambda(\mathbf{K}, \mathbf{q})/2\} \tag{4.7}$$

If the collision integral is linearized in $\Phi_\lambda(\mathbf{K}, \mathbf{q})$, it takes the form

$$\begin{aligned}
&\mathbb{C}_{\text{lin}}(\hat{b}^+_{\lambda\mathbf{K}-\mathbf{q}/2,\lambda}\hat{b}_{\lambda\mathbf{K}+\mathbf{q}/2,\lambda})\\
&\quad= \frac{i}{\hbar}\frac{\beta\pi g^2}{4V^2}\lambda \sum_{\mathbf{K}_2,\mathbf{K}_3,\mathbf{K}_4}\sum_{\lambda,\lambda_3,\lambda_4}(\mathbf{K}+\mathbf{K}_2-\mathbf{K}_3-\mathbf{K}_4)\\
&\quad\times \delta\left\{\frac{1}{\hbar}[\lambda E_\lambda(\mathbf{K},\mathbf{q}) + \lambda_2 E_{\lambda_2}(\mathbf{K}_2,\mathbf{q}) - \lambda_3 E_{\lambda_3}(\mathbf{K}_3,\mathbf{q}) - \lambda_4 E_{\lambda_4}(\mathbf{K}_4,\mathbf{q})]\right\}\\
&\quad\times \tilde{f}^0(\lambda E_\lambda(\mathbf{K},\mathbf{q}))\tilde{f}^0(\lambda_2 E_{\lambda_2}(\mathbf{K}_2,\mathbf{q}))\\
&\quad\times [1-\tilde{f}^0(\lambda_3 E_{\lambda_3}(\mathbf{K}_3,\mathbf{q}))][1-\tilde{f}^0(\lambda_4 E_{\lambda_4}(\mathbf{K}_4,\mathbf{q}))]\\
&\quad\times \left(1-\lambda\lambda_2\frac{\Delta_0^2+\xi_\mathbf{K}\xi_{\mathbf{K}_2}}{E_\mathbf{K}{}^0 E^0_{\mathbf{K}_2}}\right)\left(1-\lambda_3\lambda_4\frac{\Delta_0^2+\xi_{\mathbf{K}_3}\xi_{\mathbf{K}_4}}{E^0_{\mathbf{K}_3}E^0_{\mathbf{K}_4}}\right)\\
&\quad\times [\lambda\Phi_\lambda(\mathbf{K},\mathbf{q}) + \lambda_2\Phi_{\lambda_2}(\mathbf{K}_2,\mathbf{q}) - \lambda_3\Phi_{\lambda_3}(\mathbf{K}_3,\mathbf{q}) - \lambda_4\Phi_{\lambda_4}(\mathbf{K}_4,\mathbf{q})]
\end{aligned} \tag{4.8}$$

Note that because local energy is conserved, the condition of detailed balance holds:

$$\begin{aligned}
&\tilde{f}^0(E_\lambda(\mathbf{K},\mathbf{q}))\tilde{f}^0(E_{\lambda_2}(\mathbf{K}_2,\mathbf{q}))[1-\tilde{f}^0(E_{\lambda_3}(\mathbf{K}_3,\mathbf{q}))][1-\tilde{f}^0(E_{\lambda_4}(\mathbf{K}_4,\mathbf{q}))]\\
&\quad= [1-\tilde{f}^0(E_\lambda(\mathbf{K},\mathbf{q}))][1-\tilde{f}^0(E_{\lambda_2}(\mathbf{K}_2,\mathbf{q}))]\\
&\qquad\times \tilde{f}^0(E_{\lambda_3}(\mathbf{K}_3,\mathbf{q}))\tilde{f}^0(E_{\lambda_4}(\mathbf{K}_4,\mathbf{q}))
\end{aligned} \tag{4.9}$$

Let us now complete the linearization of Eq. (4.8) with respect to absolute equilibrium. We first introduce an expansion of the pairwise averages about absolute equilibrium:

$$\langle \hat{b}^+_{\lambda\,\mathbf{K}-\mathbf{q}/2,\lambda}\hat{b}_{\lambda\mathbf{K}+\mathbf{q}/2,\lambda}\rangle = f^0(E_\mathbf{K}{}^0) + \frac{\partial f^0(E_\mathbf{K}{}^0)}{\partial E_\mathbf{K}{}^0}\phi_\lambda(\mathbf{K},\mathbf{q}) \tag{4.10}$$

where

$$f^0(E_\mathbf{K}{}^0) = [\exp(\beta E_\mathbf{K}{}^0)+1]^{-1} = \tfrac{1}{2}[1+\tanh(\beta E_\mathbf{K}{}^0/2)] \tag{4.11}$$

If we next equate Eqs. (4.6) and (4.10), we find

$$\tilde{f}^0(E_\lambda(\mathbf{K},\mathbf{q})) + \frac{\partial\tilde{f}^0(E_\lambda(\mathbf{K},\mathbf{q}))}{\partial E_\lambda(\mathbf{K},\mathbf{q})}\Phi_\lambda(\mathbf{K},\mathbf{q}) = f^0(E_\mathbf{K}{}^0) + \frac{\partial f^0(E_\mathbf{K}{}^0)}{\partial E_\mathbf{K}{}^0}\phi_\lambda(\mathbf{K},\mathbf{q}) \tag{4.12}$$

We now expand the local equilibrium distribution about absolute equilibrium,

$$\tilde{f}^0(E_\lambda(\mathbf{K},\mathbf{q})) = f^0(E_\mathbf{K}{}^0) + \frac{\partial f^0(E_\mathbf{K}{}^0)}{\partial E_\mathbf{K}{}^0}\delta E_\lambda(\mathbf{K},\mathbf{q}) \tag{4.13}$$

where

$$\delta E_+(\mathbf{K}, \mathbf{q}) = -\delta E_{11}(\mathbf{K}, \mathbf{q}) \tag{4.14}$$

and

$$\delta E_-(\mathbf{K}, \mathbf{q}) = \delta E_{22}(\mathbf{K}, \mathbf{q}) \tag{4.15}$$

Comparing Eqs. (4.12) and (4.13), we see that

$$\left[\frac{\partial f^0(E_{\mathbf{K}}{}^0)}{\partial E_{\mathbf{K}}{}^0}\right]^{-1} \frac{\partial f^0(E_\lambda(\mathbf{K}, \mathbf{q}))}{\partial E_\lambda(\mathbf{K}, \mathbf{q})} \Phi_\lambda(\mathbf{K}, \mathbf{q}) = \phi_\lambda(\mathbf{K}, \mathbf{q}) - \delta E_\lambda(\mathbf{K}, \mathbf{q}) \tag{4.16}$$

If in Eq. (4.8) we retain contributions linear in deviations from absolute equilibrium, we obtain

$$\begin{aligned}
&\mathbb{C}_{\text{lin}}(\hat{b}^+_{\lambda\mathbf{K}-\mathbf{q}/2,\lambda}\hat{b}_{\lambda\mathbf{K}+\mathbf{q}/2,\lambda}) \\
&= \frac{i\beta\pi g^2}{\hbar V^2 4}\lambda \sum_{\mathbf{K}_2,\mathbf{K}_3,\mathbf{K}_4} \sum_{\lambda_2,\lambda_3,\lambda_4} \delta(\mathbf{K} + \mathbf{K}_2 - \mathbf{K}_3 - \mathbf{K}_4) \\
&\quad \times \delta\left[\frac{1}{\hbar}(\lambda E_{\mathbf{K}}{}^0 + \lambda_2 E^0_{\mathbf{K}_2} - \lambda_3 E^0_{\mathbf{K}_3} - \lambda_4 E^0_{\mathbf{K}_4})\right] \\
&\quad \times f^0(\lambda E_{\mathbf{K}}{}^0) f^0(\lambda_2 E^0_{\mathbf{K}_2})[1\, f^0(\lambda_3 E^0_{\mathbf{K}_3})][1 - f^0(\lambda_4, E^0_{\mathbf{K}_4})] \\
&\quad \times \left(1 - \lambda\lambda_2 \frac{\Delta_0{}^2 + \xi_{\mathbf{K}}\xi_{\mathbf{K}_2}}{E_{\mathbf{K}}{}^0 E^0_{\mathbf{K}_2}}\right)\left(1 - \lambda_3\lambda_4 \frac{\Delta_0{}^2 + \xi_{\mathbf{K}_3}\xi_{\mathbf{K}_4}}{E^0_{\mathbf{K}_3} E^0_{\mathbf{K}_4}}\right) \\
&\quad \times [\lambda\Phi_\lambda{}^0(\mathbf{K}, \mathbf{q}) + \lambda_2\Phi^0_{\lambda_2}(\mathbf{k}_2, \mathbf{q}) - \lambda_3\Phi^0_{\lambda_3}(\mathbf{k}_3, \mathbf{q}) - \lambda_4\Phi^0_{\lambda_4}(\mathbf{k}_4, \mathbf{q})]
\end{aligned} \tag{4.17}$$

where now

$$\Phi_\lambda{}^0(\mathbf{K}, \mathbf{q}) = \phi_\lambda(\mathbf{K}, \mathbf{q}) - \delta E_\lambda(\mathbf{K}, \mathbf{q}) \tag{4.18}$$

It is useful to write the collision integral in terms of a linear collision operator $\hat{C}_{\mathbf{K},\lambda}$, which is defined by the equation

$$i\hat{C}_{\mathbf{K},\lambda}\Phi_\lambda{}^0(\mathbf{K}, \mathbf{q}) = C_{\text{lin}}(\hat{b}^+_{\lambda\mathbf{K}-\mathbf{q}/2,\lambda}\hat{b}_{\lambda\mathbf{K}+\mathbf{q}/2,\lambda}) \tag{4.19}$$

As we shall see, the collision operator $\hat{C}_{\mathbf{K},\lambda}$ has a structure which enables us to decouple the kinetic equations for δv_{11} and δv_{22} into two kinetic equations, one for the bogolon spin density and another for the total bogolon density.

Let us first note that

$$\frac{\partial f_{\mathbf{K}}{}^0}{\partial E_{\mathbf{K}}{}^0} \phi_+(\mathbf{K}, \mathbf{q}, t) = -\delta v_{11}(\mathbf{K}, \mathbf{q}, t) \tag{4.20}$$

$$\frac{\partial f_{\mathbf{K}}{}^0}{\partial E_{\mathbf{K}}{}^0} \phi_-(\mathbf{K}, \mathbf{q}, t) = \delta v_{22}(-\mathbf{K}, \mathbf{q}, t) \tag{4.21}$$

Then the kinetic equation for the total bogolon density can be written

$$i\hbar \frac{\partial f_{\mathbf{K}}{}^0}{\partial E_{\mathbf{K}}{}^0} \frac{\partial \mathbf{h}(\mathbf{k}, \mathbf{q}, t)}{\partial t} - \frac{\hbar^2 \mathbf{K}\cdot\mathbf{q}}{m} \frac{\xi_{\mathbf{K}}}{E_{\mathbf{K}}{}^0} \frac{\partial f_{\mathbf{K}}{}^0}{\partial E_{\mathbf{K}}{}^0} H(\mathbf{K}, \mathbf{q}, t) = i\mathbb{C}_{\mathbf{K}}^{(+)} H(\mathbf{K}, \mathbf{q}, t) \quad (4.22)$$

where

$$h(\mathbf{K}, \mathbf{q}, t) = \phi_+(\mathbf{K}, \mathbf{q}, t) + \phi_-(-\mathbf{K}, \mathbf{q}, t) \quad (4.23)$$

and

$$\begin{aligned} H(\mathbf{K}, \mathbf{q}, t) &= h(\mathbf{K}, \mathbf{q}, t) - \delta E_{22}(-\mathbf{K}, \mathbf{q}, t) + \delta E_{11}(\mathbf{K}, \mathbf{q}, t) \\ &= h(\mathbf{K}, \mathbf{q}, \mathbf{t}) - 2\hbar \mathbf{K}\cdot\mathbf{v}_s(\mathbf{q}, t) - 2\hbar[1 + \Gamma(T)]^{-1} \frac{\xi_{\mathbf{K}}}{E_{\mathbf{K}}{}^0} \frac{\partial \phi(\mathbf{q}, t)}{\partial t} \\ &\quad + \frac{\xi_{\mathbf{K}}}{E_{\mathbf{K}}{}^0} \frac{g}{V} [1 + \Gamma(T)]^{-1} \sum_{\mathbf{K}'} \frac{\xi_{\mathbf{K}'}}{E^0_{\mathbf{K}'}} \frac{\partial f^0_{\mathbf{K}'}}{\partial E^0_{\mathbf{K}'}} h(\mathbf{K}', \mathbf{q}, t) \\ &\quad + \frac{\xi_{\mathbf{K}}}{E_{\mathbf{K}}{}^0} \chi(T)^{-1} \frac{\Delta_0}{E_{\mathbf{K}}{}^0} \sum_{\mathbf{K}'} \frac{\Delta_0}{E^0_{\mathbf{K}'}} \frac{\partial f^0_{\mathbf{K}'}}{\partial E^0_{\mathbf{K}'}} h(\mathbf{K}', \mathbf{q}, t) \end{aligned} \quad (4.24)$$

The kinetic equation for the bogolon spin density can be written

$$i\hbar \frac{\partial f_{\mathbf{K}}{}^0}{\partial E_{\mathbf{K}}{}^0} \frac{\partial m(\mathbf{K}, \mathbf{q}, t)}{\partial t} - \frac{\hbar^2 \mathbf{K}\cdot\mathbf{q}}{m} \frac{\xi_{\mathbf{K}}}{E_{\mathbf{K}}{}^0} \frac{\partial f_{\mathbf{K}}{}^0}{\partial E_{\mathbf{K}}{}^0} M(\mathbf{K}, \mathbf{q}, t) = i\mathbb{C}_{\mathbf{K}}^{(-)} M(\mathbf{K}, \mathbf{q}, t) \quad (4.25)$$

where

$$m(\mathbf{K}, \mathbf{q}, t) = \phi_+(\mathbf{K}, \mathbf{q}, t) - \phi_-(-\mathbf{K}, \mathbf{q}, t) \quad (4.26)$$

and

$$M(\mathbf{K}, \mathbf{q}, t) = m(\mathbf{K}, \mathbf{q}, t) - \frac{g}{V} \sum_{\mathbf{K}'} \frac{\partial f^0_{\mathbf{K}'}}{\partial E^0_{\mathbf{K}'}} m(\mathbf{K}', \mathbf{q}, t) \quad (4.27)$$

The collision operators $\hat{\mathbb{C}}_{\mathbf{K}}^{(+)}$ and $\hat{\mathbb{C}}_{\mathbf{K}}^{(-)}$ are defined as follows:

$$\begin{aligned} &\hat{\mathbb{C}}_{\mathbf{K}}^{(+)} H(\mathbf{K}, \mathbf{q}, t) \\ &= \frac{\beta\pi g^2}{4\hbar V^2} \sum_{\mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4} \sum_{\lambda_2, \lambda_3, \lambda_4} \delta(\mathbf{K} + \mathbf{K}_2 + \mathbf{K}_3 - \mathbf{K}_4) \\ &\quad \times \delta\left[\frac{1}{\hbar}(\lambda E_{\mathbf{K}}{}^0 + \lambda_2 E^0_{\mathbf{K}_2} - \lambda_3 E^0_{\mathbf{K}_3} - \lambda_4 E^0_{\mathbf{K}_4})\right] \\ &\quad \times f^0(E_{\mathbf{K}}{}^0) f^0(\lambda_2 E^0_{\mathbf{K}_2})[1 - f^0(\lambda_3 E^0_{\mathbf{K}_3})][1 - f^0(\lambda_4 E^0_{\mathbf{K}_4})] \\ &\quad \times \left(1 - \lambda_2 \frac{\Delta_0{}^2 + \xi_{\mathbf{K}}\xi_{\mathbf{K}_2}}{E_{\mathbf{K}}{}^0 E^0_{\mathbf{K}_2}}\right)\left(1 - \lambda_3\lambda_4 \frac{\Delta_0{}^2 + \xi_{\mathbf{K}_3}\xi_{\mathbf{K}_4}}{E^0_{\mathbf{K}_3} E^0_{\mathbf{K}_4}}\right) \\ &\quad \times [H(\mathbf{K}, \mathbf{q}, t) + \lambda_2 H(\lambda_2 \mathbf{K}, \mathbf{q}, t) - \lambda_3 H(\lambda_3 \mathbf{K}_3, \mathbf{q}, t) - \lambda_4 H(\lambda_4 \mathbf{K}_4, \mathbf{q}, t)] \end{aligned} \quad (4.28)$$

and

$$
\begin{aligned}
&\hat{\mathbb{C}}_{\mathbf{K}}^{(-)} M(\mathbf{K}, \mathbf{q}, t) \\
&\quad = \frac{\beta \pi g^2}{4\hbar V^2} \sum_{\mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4} \sum_{\lambda_2, \lambda_3, \lambda_4} \delta(\mathbf{K} + \mathbf{K}_2 - \mathbf{K}_3 - \mathbf{K}_4) \\
&\qquad \times \delta\left[\frac{1}{\hbar}(\lambda E_{\mathbf{K}}{}^0 + \lambda_2 E_{\mathbf{K}_2}^0 - \lambda_3 E_{\mathbf{K}_3}^0 - \lambda_4 E_{\mathbf{K}_4}^0)\right] \\
&\qquad \times f^0(E_{\mathbf{K}}{}^0) f^0(\lambda_2 E_{\mathbf{K}_2}^0)[1 - f^0(\lambda_3 E_{\mathbf{K}_3}^0)][1 - f^0(\lambda_4 E_{\mathbf{K}4}^0)] \\
&\qquad \times \left(1 - \lambda_2 \frac{\Delta_0{}^2 + \xi_{\mathbf{K}} \xi_{\mathbf{K}_2}}{E_{\mathbf{K}}{}^0 E_{\mathbf{K}_2}^0}\right)\left(1 - \lambda_3 \lambda_4 \frac{\Delta_0{}^2 + \xi_{\mathbf{K}_3} \xi_{\mathbf{K}_4}}{E_{\mathbf{K}_3}^0 E_{\mathbf{K}_4}^0}\right) \\
&\qquad \times [M(\mathbf{K}, \mathbf{q}, t) + M(\lambda_2 \mathbf{K}_2, \mathbf{q}, t) - M(\lambda_3 \mathbf{K}_3, \mathbf{q}, t) - M(\lambda_4 \mathbf{K}_4, \mathbf{q}, t)]
\end{aligned} \tag{4.29}
$$

We now note that the collision operator $\hat{\mathbb{C}}_{\mathbf{K}}^{(+)}$ has four eigenfunctions with eigenvalue zero. They are the three components of momentum $\mathbf{K}$ and the bogolon energy $E_{\mathbf{K}}{}^0$,

$$\hat{\mathbb{C}}_{\mathbf{K}}^{(+)} \mathbf{K} = 0, \qquad \hat{\mathbb{C}}_{\mathbf{K}}^{(+)} E_{\mathbf{K}}{}^0 = 0 \tag{4.30}$$

The collision operator $\hat{\mathbb{C}}_{\mathbf{K}}^{(-)}$ has one eigenfunction with zero eigenvalue, a constant

$$\hat{\mathbb{C}}_{\mathbf{K}}^{(-)} A = 0 \tag{4.31}$$

where A is a constant. These facts will be useful later. We note that the kinetic equation for $h(\mathbf{K}, \mathbf{q}, t)$ is still not closed because $H(\mathbf{K}, \mathbf{q}, t)$ *depends on* $\mathbf{v}_s$ and $\partial\phi/\partial t$. However, we can express these quantities in terms of $h(\mathbf{K}, \mathbf{q}, t)$.

5. CLOSURE OF THE KINETIC EQUATION

We know from thermodynamic arguments that a gradient in the chemical potential can cause the superfluid to accelerate. If we identify the time derivative of the phase with the chemical potential

$$\mu(\mathbf{q}, t) = -\hbar\, \partial\phi(\mathbf{q}, t)/\partial t \tag{5.1}$$

then Eqs. (2.7), (2.27), and (5.1) can be used to obtain

$$im\, \partial \mathbf{v}_s(\mathbf{q}, t)/\partial t = \mathbf{q}\mu(\mathbf{q}, t) \tag{5.2}$$

as we expect. We see from Eq. (2.18) that this choice is consistent. Equation (5.2) is the usual hydrodynamic equation for the superfluid velocity.

In superfluid hydrodynamics, it is always assumed that the total mass of the fluid is conserved and that there is a continuity equation for the mass density. As we can see from Eq. (4.29), bogolons are not conserved during collisions. There is no bogolon continuity equation. Furthermore, total particle number is not conserved by the bogolon kinetic equations. However, we can impose the conservation of total particle number on the system by the proper choice of the chemical potential, and in so doing we close the kinetic equations.

Variations in the total mass density can be written

$$\begin{aligned}\delta\rho(\mathbf{q}, t) &= \frac{1}{V}\sum_{\mathbf{k}} [\delta n_{22}(\mathbf{k}, \mathbf{q}, t) - \delta n_{11}(\mathbf{k}, \mathbf{q}, t)] \\ &= [1 + \Gamma(T)]^{-1}\left\{\frac{1}{V}\sum_{\mathbf{k}} \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}^0}\frac{\partial f_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}^0} h(\mathbf{k}, \mathbf{q}, t) + \frac{2\Gamma(T)}{g}\mu(\mathbf{q}, t)\right\} \qquad (5.3)\end{aligned}$$

The total momentum density operator can be written

$$\hat{\mathbf{J}} = \sum_{\lambda} \int d\mathbf{r}\, \hat{\Psi}_{\lambda}^{+}(\mathbf{r})(-i\hbar\, \nabla_{\mathbf{r}})\hat{\Psi}_{\lambda}(\mathbf{r}) \qquad (5.4)$$

From this, we derive the result

$$\begin{aligned}\mathbf{J}(\mathbf{q}, t) &= \rho\mathbf{v}_s + \frac{1}{V}\sum_{\mathbf{k}} \hbar\mathbf{k}[n_{22}(-\mathbf{k}, \mathbf{q}, t) - n_{11}(\mathbf{k}, \mathbf{q}, t)] \\ &= \rho\mathbf{v}_s + \frac{1}{V}\sum_{\mathbf{k}} \hbar\mathbf{k}\frac{\partial f_{\mathbf{k}}^0}{\partial E_k^0} h(\mathbf{k}, \mathbf{q}, t) \qquad (5.5)\end{aligned}$$

where ρ is the total particle density. The continuity equation is given by

$$i\,\partial\rho(\mathbf{q}, t)/\partial t = \mathbf{q}\cdot\mathbf{J}(\mathbf{q}, t) \qquad (5.6)$$

If we now require that Eq. (5.6) be satisfied for our fluid, we find

$$\begin{aligned}i\frac{\partial\mu(\mathbf{q}, t)}{\partial t} &= \frac{-gi}{2\Gamma(T)}\frac{1}{V}\sum_{\mathbf{k}} \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}^0}\frac{\partial f_{\mathbf{k}}^0}{\partial E_k^0}\frac{\partial h(\mathbf{k}, \mathbf{q}, t)}{\partial t} \\ &\quad + \frac{[1 + \Gamma(T)]g}{2\Gamma(T)}\left[\rho\mathbf{q}\cdot\mathbf{v}_s(\mathbf{q}, t) + \frac{1}{V}\sum_{\mathbf{k}} \hbar\mathbf{q}\cdot\mathbf{k}\frac{\partial f_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}^0} h(\mathbf{k}, \mathbf{q}, t)\right] \qquad (5.7)\end{aligned}$$

and if we use the kinetic equation (4.22) we obtain

$$\begin{aligned}i\frac{\partial\mu(\mathbf{q}, t)}{\partial t} &= -\frac{g}{2\Gamma(T)}\frac{1}{V}\sum_{\mathbf{k}} \frac{\hbar}{m}\mathbf{q}\cdot\mathbf{k}\frac{\partial f_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}^0} \\ &\quad\times\left\{\left(\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}^0}\right)^2 H(\mathbf{k}, \mathbf{q}, t) - [1 + \Gamma(T)]h(\mathbf{k}, \mathbf{q}, t)\right\} \\ &\quad + \frac{[1 + \Gamma(T)]g}{2\Gamma(T)}\rho\mathbf{q}\cdot\mathbf{v}_s(\mathbf{q}, t) - \frac{g}{2\Gamma(T)}\frac{1}{\hbar}\frac{1}{V}\sum_{\mathbf{k}} \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}^0} i\mathbb{C}_{\mathbf{k}}^{(+)}H(\mathbf{k}, \mathbf{q}, t) \qquad (5.8)\end{aligned}$$

Let us now Fourier transform the time dependence of Eq. (5.8). We write

$$\mu(\mathbf{q}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} \mu(\mathbf{q}, \omega) \tag{5.9}$$

with similar expressions for other time-dependent variables. We then can use Eqs. (4.24), (5.2), and (5.8) to solve for $\mu(\mathbf{q}, \omega)$. We will assume that $\mathbf{q}$ lies along the z axis. Then the expression for $\mu(\mathbf{q}, \omega)$ takes the following form:

$$\begin{aligned} \mu(\mathbf{q}, \omega) = \left[\Gamma(T) + R(T)\frac{q^2}{\omega^2}\right]^{-1} \Bigg(& -\frac{q}{\omega}\frac{g}{2}\frac{1}{V}\sum_{\mathbf{k}} \frac{\hbar k_z}{m}\frac{\partial f_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}^0} \\ & \times \left\{\left(\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}^0}\right)^2 - [1 + \Gamma(T)]\right\} h(\mathbf{k}, \mathbf{q}, t) \\ & - \frac{1}{\omega}\frac{g}{2\hbar\Gamma(T)}\frac{1}{V}\sum_{\mathbf{k}} \frac{\xi^{\mathbf{k}}}{E_{\mathbf{k}}^0}\, i\hat{\mathbb{C}}_{\mathbf{k}}^{(+)} h(\mathbf{k}, \mathbf{q}, t)\Bigg) \end{aligned} \tag{5.10}$$

where

$$R(T) = -g\left[\frac{\rho}{2m}[1 + \Gamma(T)] + \frac{1}{V}\sum_{\mathbf{k}} \left(\frac{\hbar k_z}{m}\right)^2 \frac{\partial f_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}^0}\right] \tag{5.11}$$

Note that only the first two terms in Eq. (4.24) contribute to Eq. (5.8). The other terms go out when the angle integrations are performed. We now have completely closed the kinetic equations. It is important to note that the term depending on the collision operator is imaginary. As we shall see in a subsequent paper, this term, unless it can be omitted, destroys the longitudinal mode.

6. HYDRODYNAMIC EQUATIONS

If we note that the collision operators $\mathbb{C}_{\mathbf{k}}^{(\pm)}$ are self-adjoint, that is,

$$\frac{1}{V}\sum_{\mathbf{k}} g(\mathbf{k})\mathbb{C}_{\mathbf{k}}^{(\pm)} h(\mathbf{k}) = \frac{1}{V}\sum_{\mathbf{k}} h(\mathbf{k})\mathbb{C}_{\mathbf{k}}^{(\pm)} g(\mathbf{k}) \tag{6.1}$$

it is a simple matter to derive the hydrodynamic equations. Let us begin with the equation for the spin density.

6.1. Spin Density

The collision operator $\hat{C}_{\mathbf{k}}^{(-)}$ has only one collisional invariant, a constant, and therefore we can derive one hydrodynamic equation from it. The spin density is defined by

$$s(\mathbf{q}, t) = \frac{1}{V}\sum_{\mathbf{k}} \frac{\partial f_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}^0} m(\mathbf{k}, \mathbf{q}, t) \tag{6.2}$$

If we sum over momentum $\mathbf{k}$ in Eq. (4.25) and use Eqs. (4.31) and (6.1), we obtain

$$i\,\partial s(\mathbf{q}, t)/\partial t = \mathbf{q}\cdot\mathbf{J}_s(\mathbf{q}, t) \tag{6.3}$$

where $\mathbf{J}_s(\mathbf{q}, t)$ is the spin current

$$\mathbf{J}_s(\mathbf{q}, t) = \frac{1}{V}\sum_{\mathbf{k}} \frac{\hbar\mathbf{k}}{m}\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}^0}\frac{\partial f_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}^0} M(\mathbf{k}, \mathbf{q}, t) \tag{6.4}$$

According to Fick's law, in the linear regime the spin current is related to the coefficient of spin diffusion D according to the equation

$$\mathbf{J}_s(\mathbf{q}, t) = -iD\mathbf{q}s(\mathbf{q}, t) \tag{6.5}$$

There is no convective contribution, because $s(\mathbf{q}, t) \to 0$ at equilibrium. If we combine Eqs. (6.3) and (6.5) and Fourier transform the time dependence [cf. Eq. (5.9)], we obtain

$$\omega = -iDq^2 \tag{6.6}$$

Thus spin waves in a simple Fermi superfluid are completely damped.

6.2. Total Density

We already have two hydrodynamic equations for quantities dependent on the total bogolon density, namely the continuity equation (5.6) and the equation for the superfluid velocity (5.2). We can obtain four more hydrodynamic equations from the kinetic equation for the total bogolon density (4.22). Let us multiply Eq. (4.22) by the momentum $\mathbf{k}$ and sum over $\mathbf{k}$. We find

$$i\frac{\partial}{\partial t}\left[\frac{1}{V}\sum_{\mathbf{k}} \hbar\mathbf{k}\frac{\partial f_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}^0} h(\mathbf{k}, \mathbf{q}, t)\right] = \mathbf{q}\cdot\left[\frac{1}{V}\sum_{\mathbf{k}} \frac{\hbar^2}{m}\,\mathbf{k}\mathbf{k}\,\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}^0}\frac{\partial f_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}^0} H(\mathbf{k}, \mathbf{q}, t)\right] \tag{6.7}$$

[cf. Eqs. (4.30) and (5.1)]. From Eqs. (5.5), (5.2), and (6.7), we obtain the following hydrodynamic equation for the momentum density $\mathbf{J}(\mathbf{q}, t)$:

$$i\,\partial\mathbf{J}(\mathbf{q}, t)/\partial t = \mathbf{q}\cdot\overline{\overline{\mathbf{\Pi}}}(\mathbf{q}, t) \tag{6.8}$$

where $\overline{\overline{\mathbf{\Pi}}}(\mathbf{q}, t)$ is the pressure tensor,

$$\overline{\overline{\mathbf{\Pi}}}(\mathbf{q}, t) = \frac{1}{V}\sum_{\mathbf{k}} \frac{\hbar^2}{m}\,\mathbf{k}\mathbf{k}\,\frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}^0}\frac{\partial f_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}^0} H(\mathbf{k}, \mathbf{q}, t) + \frac{\rho}{m}\mu(\mathbf{q}, t)\overline{\overline{U}} \tag{6.9}$$

($\overline{\overline{U}}$ is the unit tensor). Note that since the total mass density ρ can be written

$$\rho = \rho_s + \rho_n \tag{6.10}$$

and the momentum current $\mathbf{J}$ can be written

$$\mathbf{J} = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n \tag{6.11}$$

where ρ_s and ρ_n are the superfluid and normal fluid densities and $\mathbf{v}_n$ is the normal fluid velocity in the lab frame, we find from Eq. (6.5) that

$$\rho_n(\mathbf{v}_n - \mathbf{v}_s) = \frac{1}{V} \sum_{\mathbf{k}} \hbar \mathbf{k} \frac{\partial f_k^0}{\partial E_k^0} h(\mathbf{k}, \mathbf{q}, t) \tag{6.12}$$

This is the normal fluid momentum density in the superfluid rest frame.

In the linear regime the total energy can be written as a functional of the bogolon distribution and the total density. Thus, any local changes in the internal energy density of the system can be written

$$\delta U(\mathbf{q}, t) = \frac{1}{V} \sum_{\mathbf{k}} E_{\mathbf{k}}^0 [\delta v_{22}(-\mathbf{k}, \mathbf{q}, t) - \delta v_{11}(\mathbf{k}, \mathbf{q}, t)] + \mu\, \delta\rho(\mathbf{q}, t) \tag{6.13}$$

From thermodynamics, we know that

$$dU = T\, dS + \mu\, d\rho \tag{6.14}$$

where S is the total entropy per unit volume and T is the temperature. Thus, variations in the entropy density can be written

$$\delta S(\mathbf{q}, t) = \frac{1}{T} \frac{1}{V} \sum_{\mathbf{k}} E_{\mathbf{k}}^0 [\delta v_{22}(-\mathbf{k}, \mathbf{q}, t) - \delta v_{11}(\mathbf{k}, \mathbf{q}, t)] \tag{6.15}$$

Let us now multiply Eq. (4.22) by the energy $E_{\mathbf{k}}^0$ and integrate over $\mathbf{k}$. If we use Eqs. (4.30) and (6.1), we find

$$i\, \partial \delta S(\mathbf{q}, t)/\partial t = \mathbf{q} \cdot \mathbf{J}^Q(\mathbf{q}, t) \tag{6.16}$$

where $\mathbf{J}^Q(\mathbf{q}, t)$ is the entropy current and is defined by

$$\mathbf{J}^Q(\mathbf{q}, t) = \frac{1}{T} \frac{1}{V} \sum_{\mathbf{k}} E_{\mathbf{k}}^0 \frac{\hbar \mathbf{k}}{m} \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}^0} \frac{\partial f_{\mathbf{k}}^0}{\partial E_{\mathbf{k}}^0} H(\mathbf{k}, \mathbf{q}, t) \tag{6.17}$$

Equations (5.6), (5.2), (6.8), and (6.16) constitute the entire set of hydrodynamic equations for quantities involving total density in a Fermi superfluid. We now can express the currents in terms of convection contributions and transport coefficients, and we can write an equation for the hydrodynamic normal mode frequencies.

6.3. Macroscopic Dispersion Relations

The dispersion relation for the spin diffusion mode has been given in Eq. (6.6), and has a very simple form. The shear modes in a simple Fermi superfluid obey a similar dispersion relation. Let us first decompose the momentum density $\mathbf{J}$ and the normal fluid velocity $\mathbf{v}_n$ into their transverse and longitudinal parts. The longitudinal parts are defined by

$$\mathbf{J}_{\parallel} = \hat{\mathbf{q}}(\hat{\mathbf{q}}\cdot\mathbf{J}) \qquad \text{and} \qquad \mathbf{v}_{n\parallel} = \hat{\mathbf{q}}(\hat{\mathbf{q}}\cdot\mathbf{v}_n) \tag{6.18}$$

where $\hat{\mathbf{q}} = \mathbf{q}/|\mathbf{q}|$, and the transverse parts are defined by

$$\mathbf{J}_{\perp} = \mathbf{J} - \mathbf{J}_{\parallel} = \rho_n \mathbf{v}_{n\perp} \qquad \text{and} \qquad \mathbf{v}_{n\perp} = \mathbf{v}_n - \mathbf{v}_{n\parallel} \tag{6.19}$$

The superfluid velocity is completely longitudinal since it is directed along $\mathbf{q}$. There are several good references(9–11) which discuss the derivation of the linearized hydrodynamic equations for an isotropic superfluid, so we will only quote the results here. The transverse velocity obeys the equation

$$i\rho_n{}^0\, \partial\mathbf{v}_{n\perp}(\mathbf{q}, t)/\partial t = -iq^2\eta\mathbf{v}_{n\perp}(\mathbf{q}, t) \tag{6.20}$$

where η is the coefficient of shear viscosity (first viscosity), and $\rho_n{}^0$ is the equilibrium normal fluid density. The dispersion relation for the shear modes takes the form

$$\omega = -i\,q^2\eta/\rho_n{}^0 \tag{6.21}$$

Thus the shear modes are completely damped.

There are four equations which govern the propagation of the longitudinal modes. The continuity equation for total particle density is given by

$$i\,\partial\rho(\mathbf{q}, t)/\partial t = qJ_{\parallel}(\mathbf{q}, t) \tag{6.22}$$

The equation for the superfluid velocity is given by

$$\begin{aligned} i\,\partial v_s(\mathbf{q}, t)/\partial t = q\,\delta\mu(\mathbf{q}, t) - iq^2\{\zeta_3[J_{\parallel}(\mathbf{q}, t) - \rho^0 v_{n\parallel}(\mathbf{q}, t)] \\ + \zeta_4 v_{n\parallel}(\mathbf{q}, t)\} \end{aligned} \tag{6.23}$$

where ρ^0 is the total mass density of the equilibrium system, and ζ_3 and ζ_4 are second viscosities. The equation for the longitudinal part of the momentum density is given by

$$i\,\partial J_{\parallel}(\mathbf{q}, t)/\partial t = q\,\delta P(\mathbf{q}, t) - iq^2(\tfrac{4}{3}\eta + \zeta_2 - \rho^0\zeta_1)\mathbf{v}_{n\parallel} - iq^2\zeta_1 J_{\parallel}(\mathbf{q}, t) \tag{6.24}$$

where $\delta P(\mathbf{q}, t)$ denotes a fluctuation in the hydrostatic pressure and ζ_1 and ζ_2 are second viscosities. The equation for entropy density is

$$i\rho^0 \frac{\partial s(\mathbf{q}, t)}{\partial t} + is^0 \frac{\partial\rho(\mathbf{q}, t)}{\partial t} = \rho^0 s^0 q v_{n\parallel}(\mathbf{q}, t) - i\frac{K}{T^0} q^2\,\delta T(\mathbf{q}, t) \tag{6.25}$$

where K is the coefficient of thermal conductivity, s^0 is the equilibrium specific entropy, $s(\mathbf{q}, t)$ denotes a fluctuation in the specific entropy, T^0 is the equilibrium temperature, and $\delta T(\mathbf{q}, t)$ denotes a fluctuation in temperature.

In order to find the normal mode frequencies of the longitudinal modes, we must close Eqs. (6.22)–(6.25). If we choose density, temperature, and the normal and superfluid velocities as our independent variables, we can Fourier transform the time dependence of Eqs. (6.22) and write them in the form

$$\omega\,\delta\rho - q\rho_n{}^0\,\delta v_n - q\rho_s{}^0\,\delta v_s = 0 \tag{6.26}$$

$$\begin{aligned} q\left(\frac{\partial P}{\partial \rho}\right)_T^0 \delta T + q\left[\left(\frac{\partial P}{\partial \rho}\right)_T^0 - \rho^0 s^0\right]\delta T - iq^2\rho^0(\zeta_4 - \rho_n{}^0\rho_3)\,\delta v_n \\ - (\rho_s{}^0\omega + iq^2\rho_s{}^0\zeta_1)\,\delta v_s = 0 \end{aligned} \tag{6.27}$$

$$\begin{aligned} q\left(\frac{\partial P}{\partial \rho}\right)_T^0 \delta\rho + q\left(\frac{\partial P}{\partial T}\right)_\rho^0 \delta T - [\rho_n{}^0\omega + iq^2(\tfrac{4}{3}\eta + \zeta_2 - \rho_s{}^0\zeta_1)]\,\delta v_n \\ - (\rho_s{}^0\omega + iq^2\rho_s{}^0\zeta_1)\,\delta v_s = 0 \end{aligned} \tag{6.28}$$

$$\omega\left[s^0 + \rho^0\left(\frac{\partial s}{\partial \rho}\right)_T^0\right]\delta\rho + \left[\omega\frac{\rho^0 C_V{}^0}{T^0} + i\frac{K}{T^0}\right]\delta T - \rho^0 s^0 q\,\delta v_n = 0 \tag{6.29}$$

where $C_V{}^0 = T^0(\partial s/\partial T)_\rho{}^0$, $\delta\rho = \rho(\mathbf{q}, \omega)$, $\delta T = T(\mathbf{q}, \omega)$, $\delta v_s = v_s(\mathbf{q}, \omega)$, and $\delta v_n = v_{n\|}(\mathbf{q}, \omega)$. Equations (6.26)–(6.29) form a closed set of equations from which we can, in principle, obtain the dispersion relations for the normal mode frequencies.

The dispersion relation for a nondissipative superfluid takes a fairly simple form. If we set the transport coefficients equal to zero in Eqs. (6.26)–(6.29), we obtain the following equation for normal mode frequencies:

$$\omega^4 - q^2\omega^2(c_s{}^2 + u_T{}^2) + q^4 c_T{}^2 u_T = 0 \tag{6.30}$$

where c_s is the speed of adiabatic first sound

$$c_s = (\partial P/\partial \rho)_s{}^0 \tag{6.31}$$

c_T is the speed of isothermal first sound

$$c_T{}^2 = (\partial P/\partial \rho)_T{}^0 \tag{6.32}$$

and u_T is the speed of second sound when $c_s{}^2 = c_T{}^2$,

$$u_T{}^2 = \rho_s{}^0(s^0)^2 T^0/C_V{}^0\rho_n{}^0 \tag{6.33}$$

Thus, for a nondissipative superfluid, there are four undamped longitudinal modes, two first-sound modes with frequency

$$\omega_s{}^2 = \frac{q^2}{2}\left[c_s{}^2 + u_T{}^2 + (c_s{}^2 - u_T{}^2)\left(1 - \frac{4(c_T{}^2 - c_s{}^2)u_T{}^2}{(c_s{}^2 - u_T{}^2)^2}\right)^{1/2}\right] \tag{6.34}$$

and two second-sound modes with frequency

$$\omega_T{}^2 = \frac{q^2}{2}\left[c_s{}^2 + u_T{}^2 - (c_s{}^2 - u_T{}^2)\left(1 - \frac{4(c_T{}^2 - c_s{}^2)u_T{}^2}{(c_s{}^2 - u_T{}^2)^2}\right)^{1/2}\right] \tag{6.35}$$

Note that if $c_T \approx c_s$, then $\omega_s = \pm q c_s$ and $\omega_T = \pm q u_T$.

If we retain the contributions from the transport coefficients in Eqs. (6.26)–(6.29), we obtain a far more complicated quartic equation for the longitudinal mode frequencies. We will not discuss its solution here. Some approximate solutions have been given in Refs. 9–11.

7. CONCLUDING REMARKS

In the previous sections, we have obtained closed, decoupled, kinetic equations for the bogolon spin density and total bogolon density of an isotropic Fermi superfluid, and we have shown that these equations yield the usual hydrodynamic equations for an isotropic Fermi superfluid. In a subsequent paper, we will obtain approximate solutions to the linearized kinetic equations for the case of long-wavelength inhomogeneities and long time. In so doing we will obtain microscopic expressions for the amplitudes and dispersion relations for the spin diffusion mode, the shear modes, and the longitudinal modes directly from the kinetic equations without having to introduce the hydrodynamic equations. Our expressions for all the modes will include damping effects due to transport processes.

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